MATHEMATICAL SYMBOLS AND LANGUAGE

Mathematical Symbols

add	$+$	subtract	$-$
multiply	$\times \quad \bullet$	divide	$\div \quad /$
is equal to	$=$	is not equal to	$\neq$
is greater than	$>$	is less than	$<$
is greater than or equal to	$\geq$	is less than or equal to	$\leq$
is approximately equal to	$\approx$	parentheses	$(\ )$

Symbols for Multiplication

The product of 8 and 4 can be written as:

8 times 4 $\quad 8 \times 4 \quad 8 \bullet 4 \quad (8)(4) \quad \begin{array}{r} 8 \\ \times\, 4 \\ \hline \end{array}$

In algebra, we generally avoid using the × for multiplication because it could be misinterpreted as the variable x, and we cautiously use the symbol $\bullet$ for multiplication because it could be misinterpreted as a decimal point.

Symbols for Division

The quotient of 8 and 4 can be written as:

8 divided by 4 $\quad 8 \div 4 \quad 4\overline{)8} \quad \frac{8}{4} \quad 8/4$

In algebra, the preferred way to show division is with fraction notation.

Meanings for Exponents

In the expression b^n • the number b is the base • the number n is the exponent $(\text{base})^{\text{exponent}}$	$b^n = \underbrace{b \bullet b \bullet b \ldots. b \bullet b \bullet b}$ multiplied by itself n times $3^4 = 3 \bullet 3 \bullet 3 \bullet 3 = 81$

MATHEMATICAL PROPERTIES

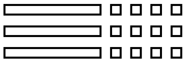
Properties of Arithmetic

Properties of arithmetic govern the manipulation of expressions. These include:

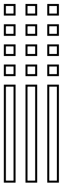
- Associative property of addition
- Commutative property of addition
- Additive identity property
- Additive inverse property
- Associative property of multiplication
- Commutative property of multiplication
- Multiplicative identity property
- Multiplicative inverse property
- Distributive property relating addition and multiplication

Does 14 × 3 Really Have the Same Value as 3 × 14?

The commutative property of multiplication asserts that the product does not depend on the order of the factors. Each of the products 3 × 14 and 14 × 3 is equal to 42.



$$\begin{array}{r} 3 \\ \times\ 14 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 14 \\ \times\ \ 3 \\ \hline \end{array}$$



Nonetheless, for some problems context is important. Although both actions require 42 marbles, the filling of 3 bags with 14 marbles each will require different supplies than the filling of 14 bags with 3 marbles each.

The Distributive Property

The distributive property relates the operations of multiplication and addition. The term “distributive” arises because the property is used to distribute the factor outside the parentheses over the terms inside the parentheses.

Suppose you earn $9.00 per hour. If you work 3 hours on Saturday and 4 hours on Sunday, one way to compute your earnings is to compute your wages for each day and then add them. Another way is to multiply the hourly wage by the total number of hours. This example illustrates the distribute property.

$$(9 \times 3) \ + \ (9 \times 4) \ = \ 9(3 + 4)$$

$$27 + 36 \ = \ 9(7)$$

WHOLE NUMBERS: MULTIPLICATION AND DIVISION

Expanded Forms of Numbers

Standard Form	Expanded Form #1	Expanded Form #2	Expanded Form #3
25	20 + 5	2(10) + 5(1)	$2(10^1) + 5(10^0)$
302	300 + 2	3(100) + 0(10) + 2(1)	$3(10^2) + 0(10^1) + 2(10^0)$

Multiplication Strategies

Skip Count	Double	Halve	Add On	Take Away
3 6 9 12 15	3 × 7 = 21, so 6 × 7 = 42 7 × 6 = 42	6 × 10 = 60, so 6 × 5 = 30 5 × 6 = 30	6 × 3 = 18 Think 19, 20, 21 so 7 × 3 = 21 3 × 7 = 21	10 × 3 = 30 30 – 3 = 27 so 9 × 3 = 27 3 × 9 = 27

Multiplication Strategies (12 x 13)

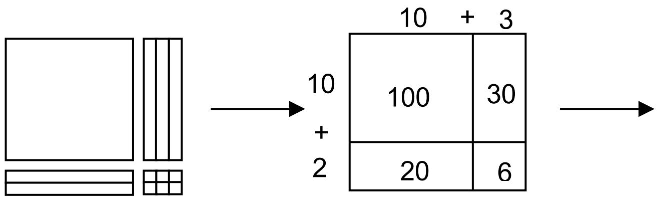
Traditional Algorithm 1

$$\begin{array}{r} 12 \\ \underline{\times\ 13} \\ 36 \\ \underline{+\ 120} \\ 156 \end{array}$$

Traditional Algorithm 2

$$\begin{array}{rl} 12 & \\ \underline{\times\ 13} & \\ 6 & = 2\times 3 \\ 30 & = 10\times 3 \\ 20 & = 10\times 2 \\ \underline{+\ 100} & = 10\times 10 \\ 156 & \end{array}$$

Area Model



$$\begin{array}{r} 100 \\ 20 \\ 30 \\ \underline{+\ \ 6} \\ 156 \end{array}$$

The Standard Division Algorithm

The standard division algorithm is an efficient process for dividing. It involves a cyclical process: divide, multiply, subtract, "bring down"… until the remainder is less than the divisor.

$14\overline{)\underline{96}\,3}$	Determine where to start	Look at the divisor. Choose digits in the dividend so that the quotient using these digits is between 1 and 9.
$14\overline{)963}$	Divide	How many 14s in 96? _____ Write this number above the 96.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ -84 \end{array}$	Multiply	Find the product of 6 and 14. Write this below the 96.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ \underline{-84} \\ 12 \end{array}$	Subtract	Find the difference between 96 and 84. Write this below the 84.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ \underline{-84\downarrow} \\ 123 \end{array}$	Bring down	Bring down the next digit.
$\begin{array}{r} 68 \\ 14\overline{)963} \\ \underline{-84\downarrow} \\ 123 \\ \underline{-112} \\ 11 \end{array}$	Divide Multiply Subtract Bring down (remainder)	Repeat the divide, multiply, subtract, bring down (if necessary) process until the remainder is less than the divisor.

Some ways to represent the dividend, divisor, quotient, and remainder:

$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}} \end{array}$ remainder

dividend = (divisor)(quotient) + remainder

$\begin{array}{r} 68 \text{ R } 11 \\ 14\overline{)963}\phantom{\text{ R } 11} \end{array}$

$\begin{array}{r} 68\,\frac{11}{14} \\ 14\overline{)963}\phantom{\,\frac{11}{14}} \end{array}$

$963 = (14)(68) + 11$

A Chunking Division Procedure

This chunking division procedure keeps the dividend intact as we “close in” on the quotient. If you do not know all your multiplication facts, this procedure may be easier than the standard division algorithm because you subtract out groups of the divisor more flexibly, but still arrive at the correct quotient. If the largest amount possible is chosen to subtract at each step, this procedure is very efficient.

Divide 761 highlighters into 3 boxes.

Step 1. Rewrite the problem.

$$\begin{array}{r|l} 3\overline{)761} & \\ & \end{array}$$

Step 2**:** Make a toolkit of multiplication facts that may be useful for this problem.

$3 \times 1 = 3$	$3 \times 10 = 30$	$3 \times 100 = 300$
$3 \times 2 = 6$	$3 \times 20 = 60$	$3 \times 200 = 600$
$3 \times 3 = 9$	$3 \times 30 = 90$	$3 \times 300 = 900$
$3 \times 4 = 12$	$3 \times 40 = 120$	$3 \times 400 = 1{,}200$

Step 3: Select a fact from the toolkit that is less than or equal to the dividend, and record.

$$\begin{array}{r|r} 3\overline{)\;761} & \\ -\underline{\;600} & 200 \\ 161 & \end{array}$$

Repeat Step 3: Continue the routine until the remainder is less than the divisor.

$$\begin{array}{r|r} 3\overline{)\;761} & \\ -\underline{\;600} & 200 \\ 161 & \\ -\underline{\;120} & 40 \\ 41 & \end{array} \longrightarrow \begin{array}{r|r} 3\overline{)\;761} & \\ -\underline{\;600} & 200 \\ 161 & \\ -\underline{\;120} & 40 \\ 41 & \\ -\underline{\;\;30} & 10 \\ 11 & \end{array} \longrightarrow \begin{array}{r|r} \multicolumn{2}{c}{253 \quad \text{R } 2} \\ 3\overline{)\;761} & \\ -\underline{\;600} & 200 \\ 161 & \\ -\underline{\;120} & 40 \\ 41 & \\ -\underline{\;\;30} & 10 \\ 11 & \\ -\underline{\;\;\;9} & \underline{+\;\;3} \\ & 253 \end{array}$$

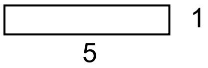
The last calculation shows that the quotient is (200 + 40 + 10 + 3) = 253, and the remainder is 2.

FACTORS AND MULTIPLES

Using Rectangles to Visualize Prime and Composite Numbers

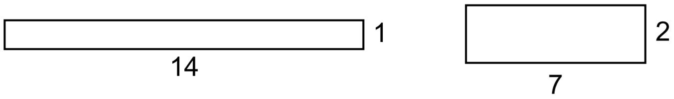
Building rectangles whose sides have whole number lengths is a geometric way to describe factors and multiples of numbers. If the area of the rectangle represents the product, then the side lengths of the rectangle represent the factors of the number.

A prime number p corresponds to only one rectangle, since p can be factored as a product in only one way, $p = 1 \bullet p$. (Here we regard the factorization $p = 1 \bullet p$ as the same as $p = p \bullet 1$, and we regard a $1 \times p$ rectangle as being the same as a $p \times 1$ rectangle.)



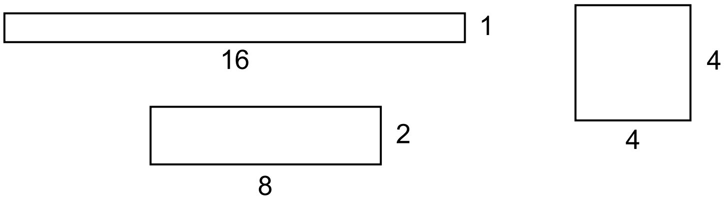
$5 = 1 \times 5$
1 and 5 are factors of 5.

A composite number n always corresponds to more than one rectangle.



$14 = 1 \times 14$
$14 = 2 \times 7$
1, 2, 7, and 14 factors of 14.

A number such as 16 is called a square number (or perfect square) because one of the rectangles it corresponds to is a square (4×4).



$16 = 1 \times 16$
$16 = 2 \times 8$
$16 = 4 \times 4$
1, 2, 4, 8, and 16 factors of 16.

Why is 1 Neither Prime nor Composite?

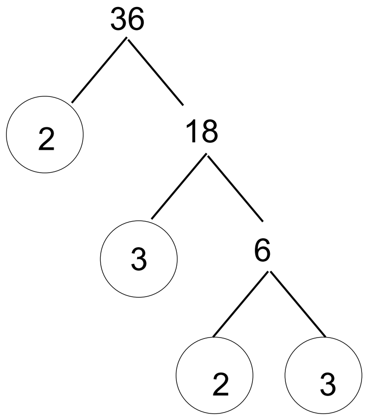
Euclid (about 300 BC) included “1” in the definition of a prime number. However, the number had to be treated as a special case in so many theorems that, by the time of Gauss (about 1800 AD), the definition was changed to exclude it.

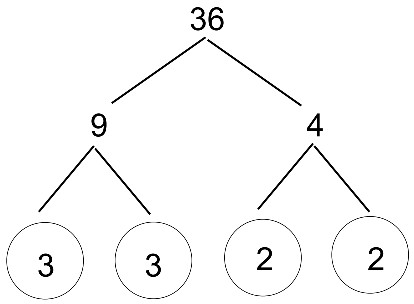
There are many definitions in mathematics that have changed over time. Originally, the definition of “rectangles” did not include “squares,” but it has become standard to include square as a subset of the rectangle family because it makes many properties easier to explain.

Factor Trees

A factor tree is a useful tool for organizing and recording the factors of a number. There may be different ways to make a factor tree for a given number, but the end result (prime factorization) will always be the same.

Here are two different factor trees to illustrate that the prime factorization of 36 is $2^2 \bullet 3^2$.





The factorization $36 = 2^2 \bullet 3^2$ is unique, except for the order of the factors.

Some Divisibility Rules

A number is divisible by…

2	if the ones digit is 0, 2, 4, 6, or 8.
3	if the sum of the digits is divisible by 3.
4	if the number represented by the last two digits is divisible by 4, or divide by 2 and then check for divisibility by 2.
5	if the ones digit is 0 or 5.
6	if it is divisible by both 2 and 3.
7	if the number formed by subtracting twice the last digit from the number formed by all digits except the last is divisible by 7.
8	if the number represented by the last three digits is divisible by 8, or divide by 2 and check for divisibility by 4.
9	if the sum of the digits is divisible by 9.
10	if the ones digit is 0.

GCF AND LCM

Greatest Common Factor (GCF)

The greatest common factor (GCF) of two numbers is the greatest factor that divides the two numbers. Here are three different ways to find the GCF of two numbers.

Tensaye has 12 bottles of water and 18 granola bars. She wants to use them to make care packages for the homeless. How many care packages can Tensaye make so that there are the same number of bottles of water and granola bars in each package?

Method 1: Use a list to find the GCF of 12 and 18

List all the factors of 12: 1, 2, 3, 4, 6, and 12.

List all the factors of 18: 1, 2, 3, 6, 9, and 18.

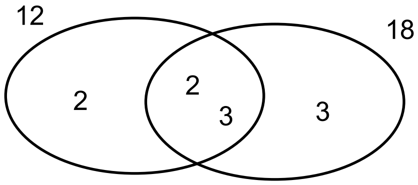
We can see that the factors 1, 2, 3, and 6 appear in both lists. Since 6 is the greatest factor from both lists that divides 12 and 18, the greatest common factor (GCF) of 12 and 18 is 6.

Method 2: Use a Venn Diagram to find the GCF of 12 and 18

Write each number as a product of primes.

$12 = 2 \bullet 2 \bullet 3$ and $18 = 2 \bullet 3 \bullet 3$

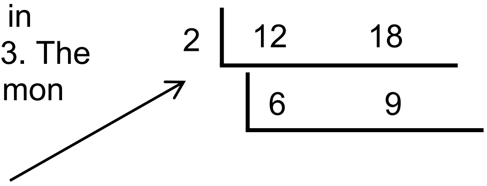
Write all the prime factors of 12 and 18 in a Venn Diagram, including overlapping factors. The product of the prime factors in the overlap is 6, so the GCF of 12 and 18 is 6.



Method 3: Use "repeated division" to find the GCF of 12 and 18

Divide each number by any common factor greater than 1. In this case, we can begin by dividing both numbers by 2. The resulting quotients are 6 and 9.

Keep dividing until both resulting quotients have no factors in common greater than 1. In this case, we can still divide by 3. The resulting quotients are now 2 and 3, and they have no common factors greater than 1.



The GCF is the product of the factors along the side. Therefore, the GCF of 12 and 18 is 6.

Since the GCF of 12 and 18 is 6, Tensaye can make 6 care packages for the homeless.

Least Common Multiple (LCM)

The least common multiple (LCM) of two numbers is the least number that is a positive multiple of both numbers. Here are three different ways to find the LCM of two numbers.

Tensaye wants buy bottles of water and granola bars to make care packages for the homeless. Bottle of water come in packages of 12, and granola bars are sold in packages of 18. How many bottles of water and how many granola bars should Tensaye buy so that she has the same number of each item?

Method 1: Use a list to find the LCM of 12 and 18

The multiples of 18 are: 18, 36, 54, 72, 90, 108, 126, 144, 162, 180, ...

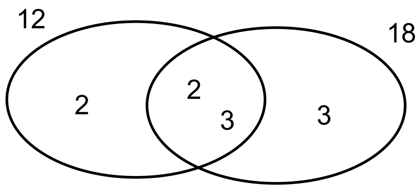
The multiples of 12 are: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120,… .

The multiples that 12 and 18 have in common are 36, 72, 108, ... We can see that 36 is the least multiple the two numbers have in common. Therefore, the LCM of 12 and 18 is 36.

Method 2: Use a Venn Diagram to find the LCM of 12 and 18

Write each number as a product of primes.

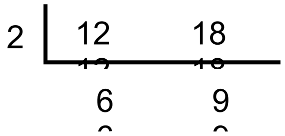
$12 = 2 \bullet 2 \bullet 3$ and $18 = 2 \bullet 3 \bullet 3$



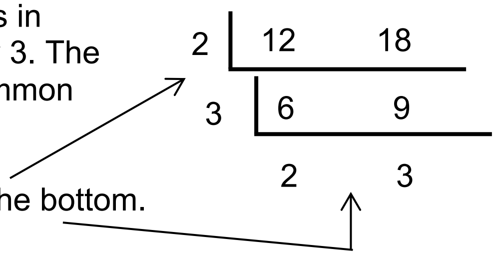
Write all the prime factors of 12 and 18 in a Venn Diagram, including overlapping factors. The product of all the prime factors in the diagram is 36, so the LCM of 12 and 18 is 36.

Method 3: Use "repeated division" to find the LCM of 12 and 18

Divide each number by any common factor greater than 1. In this case, we can begin by dividing both numbers by 2. The resulting quotients are 6 and 9.



Keep dividing until both resulting quotients have no factors in common greater than 1. In this case, we can still divide by 3. The resulting quotients are now 2 and 3, and they have no common factors greater than 1.



The LCM is the product of the factors along the side and the bottom. Therefore, the LCM of 12 and 18 is 36.

Since the LCM of 12 and 18 is 36, Tensaye should buy 36 bottles (or 3 packages) of water and 36 granola bars (or 2 packages) so that she has the same number of each item.

ORDER OF OPERATIONS

Order of Operations

There are many mathematical conventions that enable us to interpret mathematical notation and to communicate efficiently about common situations. The agreed-upon rules for interpreting mathematical notation, important for simplifying arithmetic and algebraic expressions, are called the order of operations.

Step 1: Do the operations in grouping symbols first (e.g., use rules 2-4 inside parentheses).

Step 2: Calculate all the expressions with exponents.

Step 3: Multiply and divide in order from left to right.

Step 4: Add and subtract in order from left to right.

Example:

$$\frac{3^2 + (6 \bullet 2 - 1)}{5} = \frac{3^2 + (12 - 1)}{5} = \frac{3^2 + (11)}{5} = \frac{9 + (11)}{5} = \frac{20}{5} = 4$$

There are many times for which these rules make complete sense and are quite natural. Take this case, for example:

You purchase 2 bottles of water for $1.50 each and 3 bags of peanuts for $1.25 each. Write an expression for this situation, and simplify the expression to find the total cost.

Expression:

$$\underbrace{2 \bullet (1.50)}_{3.00} + \underbrace{3 \bullet (1.25)}_{3.75} = 6.75$$

In this problem it is natural to find the cost of the 2 bottles of water and then the cost of the 3 bags of peanuts prior to adding these amounts together. In other words, we perform the multiplication operations before the addition operation.

Note however that if we were to perform the operations in order from left to right (as we read the English language from left to right), we would obtain a different result:

$$2\,(1.50) = 3 \rightarrow 3 + 3 = 6 \rightarrow 6\,(1.25) = 7.50$$

Using Order of Operations to Simplify Expressions

Order of Operations	**Example:** $\frac{2^3 \bullet 3(4-2)}{4+2 \bullet 10}$	**Comments**
1. Simplify expressions within grouping symbols.	$\frac{2^3 \bullet 3(2)}{4+2 \bullet 10}$	Parentheses are grouping symbols, and 4 – 2 = 2. The fraction bar, used for division, is also a grouping symbol, so the numerator and denominator must each be simplified completely prior to dividing.
2. Calculate powers and roots.	$\frac{8 \bullet 3(2)}{4+2 \bullet 10}$	$2^3 = 2 \bullet 2 \bullet 2 = 8$
3. Perform multiplication and division from left to right.	$\frac{24 \bullet 2}{4+20} = \frac{48}{4+20}$	
4. Perform addition and subtraction from left to right.	$\frac{48}{24} = 2$	The groupings in the numerator and denominator have been simplified, so the final division can be performed.

Heads Up!

Step 3 on the previous page instructs us to perform multiplication before division.

RIGHT!	WRONG!
$8 \div 4 \bullet 2$ $= 2 \bullet 2$	$8 \div 4 \bullet 2$ $\neq 8 \div 8$

Step 4 on the previous page instructs us to perform addition before subtraction.

RIGHT!	WRONG!
$14 - 6 + 2$ $= 8 + 2$	$14 - 6 + 2$ $\neq 14 - 8$

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