TEACHER EDITION

MathLinks GRADE 8

COMPILED MATH BACKGROUND

MATH BACKGROUND

Why Does a Circle Have 360 Degrees?

Lack of sources makes it impossible to determine how various enumeration systems originated. There has been a great deal of speculation about why various systems were devised and how they interacted.

We know virtually nothing about the systems of enumeration used by the Sumerian civilization in Mesopotamia (before say 2200 BCE). However, we have extensive evidence of the systems of enumeration used by the subsequent old Babylonian civilization, going back past the Code of Hammurabi (1800 BCE). The evidence comes from excavations of mathematical cuneiform tablets with extensive records of warehouse inventories and also astronomical observations and calculations.

The two most important features of Babylonian mathematics are:

1. a place value system of recording numbers, and
2. use of a base-60 system of enumeration.

The use of a base-60 system is convenient because 60 has so many divisors. One theory about how this system evolved is based on currency conversions. In antiquity there were numerous currency systems, with different bases (think of pounds, shillings, and pence). It might have been natural for some merchants in trading centers to adopt a unit that would facilitate conversion rates among various currencies.

Whatever the origin of the base-60 system, the choice was serendipitous in the sense that it facilitated many mathematical calculations, such as calculations of reciprocals. In the golden era of Mesopotamian astronomy (final four centuries BCE), the Babylonian astronomers had a big advantage over the Egyptians, who were tied to a system of unit fractions that did not facilitate the intricate calculations required by astronomers.

The number 360 of degrees in a circle is a relatively recent development, stemming from the golden age of Mesopotamian astronomy. One explanation for the number 360 is that it might have been natural to approximate a circle (think in terms of the constellations through which the moon and planets pass) by chords that are sides of a regular hexagon composed of 6 equilateral triangles, thereby dividing the skies into 6 sectors. In a base-60 system, it would be natural to divide further the central angle of each triangle into 60 degrees, thereby leading to a total of 6 times 60, or 360, degrees in the circle. Further subdivision leads to 60 minutes in a degree, and 60 seconds in a minute.

A convenient property of the number 360 is that it is close to the number of days in a calendar year. The sun advances across the firmament by close to $\frac{1}{360}$ of a turn each day. The Egyptian calendar, claimed to have been adopted before 4000 BCE, divided the year into 12 months of 30 days for a total of 360 days, plus 5 extra days added to the end of the cycle. This scheme was modified by the Romans (Julian calendar of 12 months with a total of 365 days), and it was fine-tuned by Pope Gregory XIII (Gregorian calendar, with a 400-year cycle of prescribed leap years).

Angle Measurement

In Euclidean geometry, an <u>angle</u> is a geometric shape formed by two (distinct) rays in a half-plane that share a common endpoint (the vertex of the angle). Through a limiting process, it is possible to assign degree measures to angles (in a half-plane) so that:

1. the measure of an angle is a positive number between 0 degrees and 180 degrees;
2. a right angle has measure 90 degrees;
3. when we bisect an angle, we get adjacent angles whose measures are both half of the original angle; and
4. when we adjoin two adjacent angles in a half-plane, we get an angle whose measure is the sum of the measures of the two angles.

The degree measure of an angle can be expressed in terms of the length of the minor circular arc cut out by the angle on a circle centered at its vertex. In fact, the <u>degree measure of the angle</u> is 360 times the fraction of the length of the circle cut out by the angle.

Example: Since a right angle cuts out a quarter of a circular arc centered at the vertex, its degree measure is a quarter of 360, or 90 degrees.

Closely related to the measure of Euclidean angles is the angle of rotation of a ray as it turns about its vertex. If we rotate a ray through $\frac{1}{360}$ of a circle in a counterclockwise direction, we create an angle that measures 1 degree. We say that the angle of rotation is 1 degree. Similarly, if we rotate a ray counterclockwise through $\frac{2}{360}$ of a circle, we create an angle of measure 2 degrees, and we say that the angle of rotation is 2 degrees. And so on. Continuing in this fashion, we are led to angles of rotation of more than 180 degrees.

Example: The counterclockwise rotation of a ray through $\frac{3}{4}$ of a circle corresponds to an angle of rotation of 270 degrees (270 = $\frac{3}{4}$ times 360). The counterclockwise rotation of a ray through one full circle, bringing it back to its starting position, corresponds to an angle of rotation of 360 degrees (360 = 1 times 360). The counterclockwise rotation of a ray through two full circles (twice around), bringing it back to its starting position, corresponds to an angle of rotation of 720 degrees (720 = 2 times 360).

The angle of rotation can also be negative. A negative angle of rotation corresponds to rotating the ray in the negative direction, that is, in the clockwise direction.

Example: If we rotate the ray a quarter of a turn clockwise, we create a right angle, but since the sense of the rotation is “backwards,” the angle of rotation is -90 degrees instead of +90 degrees.

Unlike the degree measures in Euclidean geometry, which run between 0 and 180 degrees, the angles of rotation run the entire number line, from $-\infty$ to $+\infty$.

A fundamental property of angles of rotation is the <u>angle addition principle</u>: The result of rotating a ray by c degrees, and then rotating it by d degrees, is a rotation of the ray by $c + d$ degrees.

Example: A rotation of a ray by 90 degrees, followed by another rotation of the ray by 90 degrees, results in a rotation of the ray by 180 degrees (90 + 90 = 180). The ray ends up pointing in the opposite direction of its initial position.

The Parallel Postulate

The parallel postulate is historically the most interesting postulate in Euclid's *Elements*. There are many equivalent formulations of the parallel postulate. Euclid's original formulation of the postulate is rather complicated:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side with the angles less than the two straight angles.

Recall that two lines in the plane are parallel if they do not meet. Perhaps the most popular textbook formulation of the parallel postulate is the following:

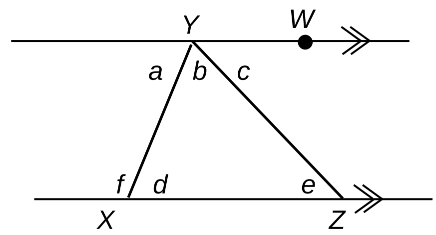
> Given a straight line and a point not on it, there exists one and only one straight line through the point and parallel to the given line.

Mathematicians tried for centuries without success to prove that the parallel postulate is a consequence of Euclid's other axioms. Finally, in the 1800's, through the work of a several mathematicians (Gauss, Bolyai, Lobachevsky, Beltrami), a geometry was discovered that satisfies all of Euclid's other axioms, but for which the parallel postulate fails. That geometry is called hyperbolic geometry.

In Euclidean geometry, the sum of the measures of the angles of a triangle is equal to 180° (a straight angle). It turns out that in hyperbolic geometry, the sum of the measures of the angles of a triangle is always less than 180°. In contrast, in spherical geometry (which does not satisfy all of Euclid's other postulates) the sum of the measures of the angles of a triangle is always greater than 180°.

Angle Sums in Triangles

Here are two important facts about angle sums in triangles.
They can be proved based on the properties of angles formed when a transversal cuts two parallel lines.



1. The sum of the measures of the angles in a triangle is equal to 180 degrees.

   Symbolically: $|\angle b| + |\angle d| + |\angle e| = 180^\circ$

Statements	Reason
There is a line $\overleftrightarrow{WY}$ passing through Y and parallel to $\overleftrightarrow{XZ}$.	Parallel postulate
$|\angle a| + |\angle b| + |\angle c| = 180^\circ$	Sum of measures of angles on a straight line equals 180°.
$|\angle a| = |\angle d|$ and $|\angle c| = |\angle e|$	If two lines are parallel, then alternate interior angles have equal measures.
$|\angle b| + |\angle d| + |\angle e| = 180^\circ$	Substitution.

2. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent interior angles.

   Symbolically: $|\angle b| + |\angle e| = |\angle f|$

Statements	Reason
$180^\circ = |\angle f| + |\angle d|$	The sum of the measures of supplementary angles is 180 degrees.
$|\angle b| + |\angle d| + |\angle e| = |\angle f| + |\angle d|$	Substitution. The sum of the measures of the angles in a triangle is equal to 180 degrees.
$|\angle b| + |\angle e| = |\angle f|$	Addition property of equality.

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Approximating Square Roots by Linear Interpolation

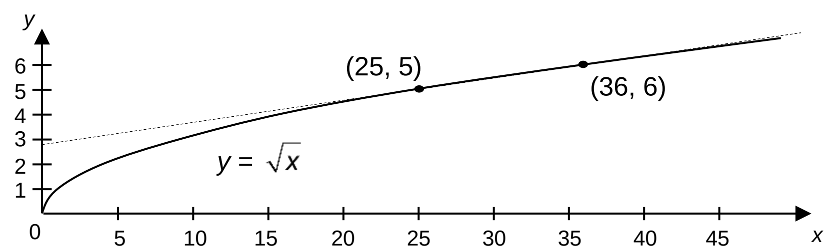
Linear interpolation is a method by which the values $y = f(x)$ of a function f on an interval $x_1 < x < x_2$ are estimated by the values of the linear function $y = mx + b$ that matches the values of f at the endpoints of the interval. The parameters m and b satisfy the two equations $f(x_1) = mx_1 + b$ and $f(x_2) = mx_2 + b$. The graph of the linear approximation $y = mx + b$ is then the straight line segment joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f.

The linear approximation can be found directly through proportional reasoning, without writing down any equations of lines. To illustrate, we approximate values of $\sqrt{x}$.

To find an approximate value for $\sqrt{27}$:

First find the closest perfect square that is less than 27 and the closest perfect square that is greater than 27. Then $25 < 27 < 36$. The points 25 and 36 will be the endpoints of the interval of interpolation.

Take the square root of each number: $\sqrt{25} = 5$, $\sqrt{36} = 6$. We aim to approximate $\sqrt{27}$ using the y-coordinate of the straight line through (25, 5) and (36, 6).



Now, 27 is 2 units larger than 25, and 36 is 11 units larger than 25. Thus 27 is two elevenths $\left(\frac{2}{11}\right)$ of the distance from 25 to 36. We approximate $\sqrt{27}$ by the number that is two elevenths of the distance from $\sqrt{25}$ to $\sqrt{36}$, that is, two elevenths of the distance from 5 to 6 (proportional reasoning!!). This number is $5 + \frac{2}{11} = 5\frac{2}{11}$.

Summary: On a number line, 27 is $\frac{2}{11}$ of the distance from 25 to 36. Therefore, $\sqrt{27}$ is approximately equal to $\frac{2}{11}$ of the distance from $\sqrt{25}$ to $\sqrt{36}$, which is $5 + \frac{2}{11}$. The approximation of $\sqrt{27}$ is $5\frac{2}{11}$.

By linear interpolation to the nearest thousandth: $5\frac{2}{11} \approx 5.182$

Accurate to the nearest thousandth: $\sqrt{x} = 5.196$

The Principal Square Root

Every positive number a has two square roots. One of them, denoted by $\sqrt{a}$, is positive. It is referred to as the principal square root of a. The other square root of a is $-\sqrt{a}$, which is negative. Together, these two roots are denoted $\pm\sqrt{a}$.

Example: The principal square root of 9 is 3, that is, $\sqrt{9} = 3$. The other square root of 9 is -3.

Danger! The principal square root of x^2 is not necessarily x. If $x < 0$, the principal square root of x^2 is $-x$.

For a formula for finding the principal square root of a squared variable that is valid for all values of the variable, we use absolute values:

$$\sqrt{x^2} = |x|$$

This formula holds for all values of x, including negative values and zero.

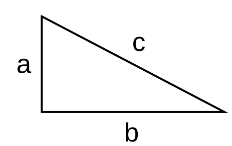
Example: $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$.

Statements of the Pythagorean Theorem

Important theorems often have slogan forms. Slogan forms are useful in that they are easy to remember and they trigger recalling the full statement of the theorem. However, the slogan form of a theorem should not be confused with a correct statement of the theorem.

The slogan form of the Pythagorean theorem is:

a squared plus *b* squared equals *c* squared
or
$a^2 + b^2 = c^2$.



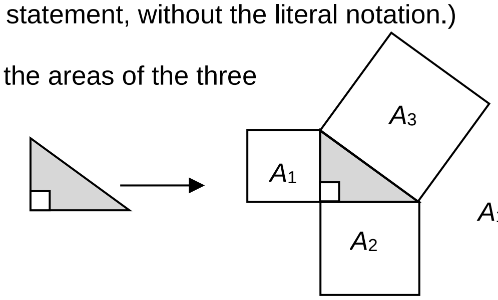
While easily remembered, this statement is at best an incomplete statement of the Pythagorean theorem. There is no reference to a right triangle nor identification of variables. Here are two more complete statements of the Pythagorean theorem:

1. For a right triangle with legs of lengths a and b and hypotenuse of length c, a squared plus b squared equals c squared. (This is a precise statement of the theorem. It has an algebraic flavor, as derived in the cut-up proof of the Pythagorean theorem.)

2. For a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse. (This is a rewording of the first statement, without the literal notation.)

Euclid stated the theorem more geometrically, in terms of the areas of the three squares sharing sides with the triangle.

3. For a right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs.



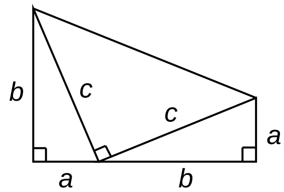
$A_1 + A_2 = A_3$

Regardless of the statement of the Pythagorean theorem preferred, it is useful to keep the slogan form in immediate access memory, where it can be used to kick-start your recollection of the circle of ideas surrounding the Pythagorean theorem.

President Garfield's Proof of the Pythagorean Theorem

The proof explored in this lesson is only one of more than 400 proofs of the Pythagorean theorem that have been recorded. Many of the proofs are simple variations of another proof.

Here is a proof discovered in 1876 by President James A. Garfield while a member of the House of Representatives. Garfield was also a mathematics teacher.



Let A be the area of the entire figure, which is a trapezoid. The area of the trapezoid is

$$A = \frac{(a+b)(a+b)}{2} = \frac{a^2 + 2ab + b^2}{2} .$$

Since the area of the trapezoid is the sum of the areas of the three triangles, we have

$$A = \frac{1}{2}(ab) + \frac{1}{2}c^2 + \frac{1}{2}(ab) = \frac{c^2 + 2ab}{2} .$$

Equating the two expressions for A, we obtain

$$\frac{a^2 + 2ab + b^2}{2} = \frac{c^2 + 2ab}{2} .$$

Hence, $a^2 + b^2 = c^2$.

When Is a Proof a Proof?

The investigation where students establish for the formula for the area of a circle is certainly convincing. Is it a proof? Yes and no. It does not constitute a formal proof, yet it contains the essential line of reasoning of a correct proof. A person with a sophisticated background in mathematical analysis (epsilons and deltas) may look at the argument and say that yes, it is a proof, in the sense that he or she can translate to the language of the analyst to form a correct formal proof.

Even mathematicians rarely produce complete formal proofs of their theorems. Hyman Bass suggests:

> "Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists."

This statement sheds light on how we might treat mathematical reasoning and proof in the early grades—what we might expect, and what we should not expect.

In any event, the idea of proof is a flexible notion. The conventions and standards of the mathematical community—whether that community is the group of students in a classroom or the readers of a mathematics research journal—play a role in deciding whether an argument is a valid proof.

"Mathematics and Teaching," in *I, Mathematician*, edited by P. Cassazza, S,G.Krantz, and R.D. Ruden, MAA, 2015

The Converse of a Theorem

The converse of the statement “A implies B” is the statement “B implies A.” In other words, the converse of the statement “if A then B” is the statement “if B then A.” Similarly, the converse of the statement “all aardvarks are mammals” is the statement “all mammals are aardvarks.”

The converse of a true mathematical statement may or may not be true. It is often of interest to phrase a mathematical theorem so that it has a natural converse statement and then to determine whether that converse statement is true or whether it is false.

Example 1:

statement: If n is an even integer, then $n + 1$ is an odd integer. (TRUE)

converse: If $n + 1$ is an odd integer, then n is an even integer. (TRUE)

Example 2:

statement: If n is a divisible by 9, then n is divisible by 3. (TRUE)

converse: If n is a divisible by 3, then n is divisible by 9. (FALSE)

Example 3:

statement: If the triangle T is a right triangle, then the sum of the squares of the lengths of the two shorter sides is equal to the square of the length of the longer side. (TRUE)

converse: If the sum of the squares of the lengths of the two shorter sides of a triangle T is equal to the square of the length of the longer side, then T is a right triangle. (TRUE)

Note that this last statement and its converse are worded so that it is clear that the statements concern the world of triangles. The statement about triangles is the Pythagorean theorem, which is true. The converse of the Pythagorean theorem, as stated, is also true.

The converse of the Pythagorean theorem is useful. Carpenters can check that an angle is a right angle by laying out sides of length 3 and 4 from the vertex and measuring the length of the third side of the triangle so formed. The angle is a right angle if and only if the third side has length 5.

Decimal Expansions of Rational Numbers

There are three things to know about decimal expansions of rational numbers.

1. Rational numbers have decimal expansions that repeat, either with a repeating pattern of nonzero digits, or with zeros from some point on (terminating decimals).

   The reason behind this is that the decimal expansion of a rational number (quotient of integers) can be obtained by the standard algorithm for division (long division), and since there are only a finite number of possibilities for the remainder, the sequence of steps must repeat after some point. If the divisor is n, then there are only n possibilities for the remainder, and eventually the process must repeat at or before the n^{th} step. For example, if the divisor is 7, the only possibilities for the remainder are 0, 1, 2, 3, 4, 5, 6.

2. Every repeating decimal is the decimal expansion of a rational number.

   There is a clever procedure for converting repeating decimals to fractions. We illustrate this by converting the decimal x = 0.16666… to a quotient of integers. Notice the unconventional order in writing down the steps. This is done to simplify the arithmetic for the students.

$10x = 1.66666...$	(2)	Notice that step 2 is above step 1. • The clever trick is to multiply both sides of the equation in step 1 by a power of 10 that will "line up" the repeating portion of the decimal. • Then subtract the expressions in step 1 from step 2. This will make the repeating portion equal zero (step 3). • Finally, solve for x and simplify your result into a quotient of integers (step 4).
Let $x = 0.16666...$	**(1)**	
$9x = 1.5$	(3)	
$x = \frac{1.5}{9} = \frac{15}{90} = \frac{1}{6}$	(4)	

   The procedure for converting a decimal to a rational number gives a different rational number for each decimal except in the special case that the decimal is terminating or is repeating with all 9's. For instance, it shows that 0.29999…. represents the same rational number as 0.3.

   Students are often surprised that 0.9999… is a decimal representation of 1, as can be seen by using this procedure to convert 0.9999… to a quotient of integers. One way to look at this is to think of decimal expansions as being "addresses" of real numbers. Each irrational number, and each rational number that cannot be represented by a finite decimal, has a unique address, posted by the front door of the location. However, rational numbers with terminating decimal expansions have two addresses, a terminating address on the front door, and an address ending in all 9's on a side door.

3. The rational numbers with terminating decimal expansions are the rational numbers that can be expressed in the form $r = m/n$, where the only (possible) prime factors of n are 2 and 5.

   Any terminating decimal r with q digits to the right of the decimal point can be expressed in the form $r = k/10^q$. When this fraction is simplified, the only prime factors of the denominator are 2 and 5. If, on the other hand, the only prime factors of the denominator n are 2 and 5, a multiple of n will have the form 10^q, and if we multiply the numerator m by the same factor, we obtain an equivalent fraction of the form $k/10^q$. For example,

$$\frac{3}{40} = \frac{3}{(2^3)(5)} = \frac{(3)}{(2^3)(5)} \bullet \frac{5^2}{5^2} = \frac{75}{100} = 0.075$$

Subsets of the Real Numbers

Here are the standard notations for the most important subsets of the set of real numbers.

Natural numbers: *N* = {1, 2, 3, …}

Whole numbers: *W* = {0, 1, 2, 3, …}

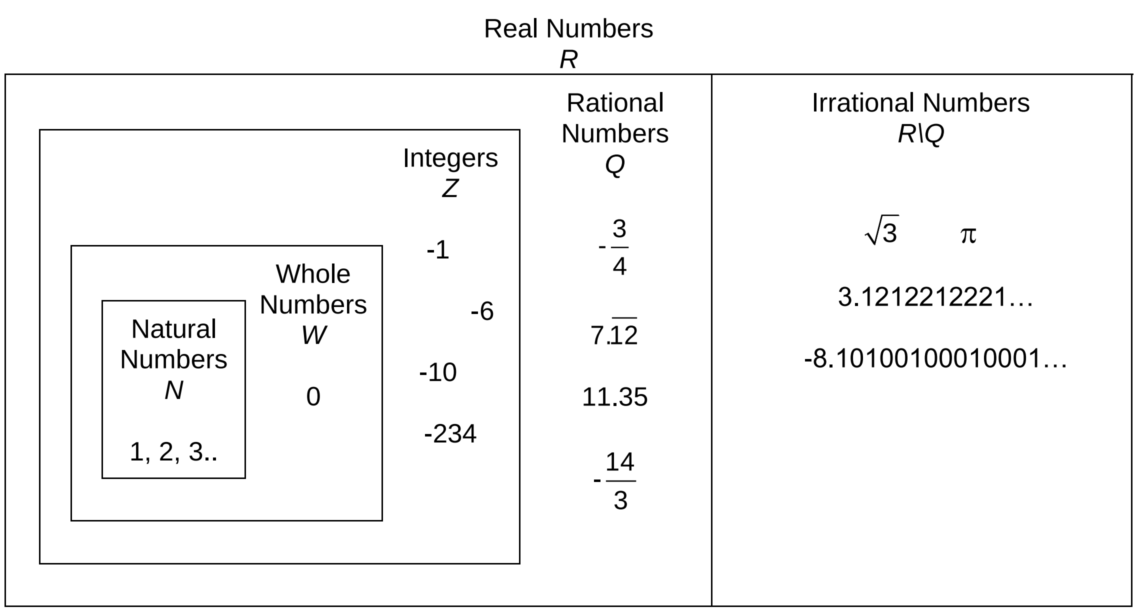
Integers: *Z* = {…, -3, -2, -1, 0, 1, 2, …} Mathematicians use the letter *Z* to denote the integers. The letter *Z* comes from the German word "Zahl", which means "number."

Rational numbers: *Q* = quotients of integers = {*m*/*n*: *m* and *n* integers, $n \neq 0$}. When rational numbers are written as decimals, they either terminate or repeat. The letter *Q* denoting rational numbers comes from the word "quotient."

Real numbers: The letter *R* denotes the set of real numbers. The real numbers may be described as numbers that can be represented by finite or infinite decimals. Rational numbers correspond to *repeating* decimal expansions (some of which repeat in zeros, or terminate), and real numbers correspond to ALL decimal expansions. The real numbers are sometimes referred to as the *real number line,* which may be represented pictorially.

Irrational numbers: Irrational numbers are simply real numbers that are not rational. Their decimal approximations do not repeat. When we think of the rational numbers as lying on a real number line, we may think of the real numbers as an extension of the rational numbers obtained by filling in "holes" corresponding to irrational numbers. There are infinitely many irrational numbers in the real number system, and surprisingly enough, there are more irrational numbers than rational numbers; this is known as the density of the irrationals in the real numbers. There is no fixed notation for irrational numbers. However, one notation for the set of irrational numbers is *R**Q*, which is interpreted as the real numbers with the rational numbers excluded. Another is *IR.*

With these notations, we have: $N \subset W$, $W \subset Z$, $Z \subset Q$, $Q \subset R$



MATH BACKGROUND

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Example: The principal square root of 9 is 3, that is, $\sqrt{9} = 3$. The other square root of 9 is -3.

Danger! The principal square root of x^2 is not necessarily x. If $x < 0$, the principal square root of x^2 is $-x$.

For a formula for finding the principal square root of a squared variable that is valid for all values of the variable, we use absolute values:

$$\sqrt{x^2} = |x|$$

This formula holds for all values of x, including negative values and zero.

Example: $\sqrt{(-3)^2} = \sqrt{9} = 3$.

Place Value Names

In the base-10 number system, each place has a value ten times that of the place value to its right and one-tenth the value of the place to its left.

thousands	hundreds	tens	ones	tenths	hundredths	thousandths	ten thousandths

To determine what a digit stands for in a number, multiply the digit by its place value. For example, in the numeral 345.67, the 4 stands for 4(10) = 40, and the 7 stands for $7\left(\frac{1}{100}\right) = 0.07$.

Another way to interpret the meaning of a digit to the right of the decimal is to divide by the whole number part of the place value name. For example, in the numeral 345.67, the meaning of the 7 is $7 \div 100 = \frac{7}{100}$.

Simplifying Radical Expressions

There is no standard form for the "simplified" version of a radical expression. The simplified form should be easy to grasp and easy to use in whatever application we have in mind for it. For instance, the radical expression $\sqrt{\frac{14{,}400}{243}}$ is a rather imposing and ugly number. It might be difficult to have an idea at a glance how large a value it is or to place it on a number line. So, we seek to simplify it using the multiplication and division properties of square roots:

$$\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}, \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \text{ for } b \neq 0$$

Here are some guidelines for simplification:

1. Remove perfect squares from under the radical, when possible.

Example 1: $\sqrt{25} = 5$

Example 2: $\sqrt{75} = \sqrt{25 \bullet 3} = \sqrt{25} \bullet \sqrt{3} = 5\sqrt{3}$

2. Remove quotients from under the radical.

Example 3: $\sqrt{\frac{9}{4}} = \frac{3}{2}$

Example 4: $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$ or $\frac{1}{2}\sqrt{5}$

3. In a quotient of radical expressions, remove radicals from the denominator.

Example 5: $\sqrt{\frac{3}{5}} = \sqrt{\frac{3}{5}}\sqrt{\frac{5}{5}} = \sqrt{\frac{15}{25}} = \frac{\sqrt{15}}{5}$ or $\frac{1}{5}\sqrt{15}$

Multiplying by 1 in the form of $\sqrt{\frac{5}{5}}$ is called "rationalizing the denominator."

If we apply these principles to the radical $\sqrt{\frac{14{,}400}{243}}$, we obtain

$$\sqrt{\frac{14{,}400}{243}} = \frac{\sqrt{14{,}400}}{\sqrt{243}} = \frac{\sqrt{(12^2)(10^2)}}{\sqrt{(9^2)(3)}} = \frac{(12)(10)}{9\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{(12)(10)\sqrt{3}}{(9)(3)} = \frac{40\sqrt{3}}{9}$$

The expression $\frac{40\sqrt{3}}{9}$ is much prettier than the original expression. Since $\frac{40}{9}$ is a little greater than 4, this expression shows us that the number is a little greater than $4\sqrt{3}$, or about 7.

MATH BACKGROUND

The Obvious Rule May Not Be the Only Rule

If only a finite number of terms of a sequence are given, and no other additional information is provided, it is always possible to find infinitely many (infinite) sequences that have the given terms.

Suppose that the first terms of a sequence are 4, 6, 8, 10, … . This suggests that each term in the sequence is obtained by adding 2 to the preceding term, that is, $a_1 = 4$ and $a_{n+1} = a_n + 2$ for $n > 1$. This may be the simplest sequence pattern with the given initial terms. But there are many other sequences that begin with the same four terms. For instance, the rule $a_1 = 4$ and $a_{n+1} = a_n + 2 + (n – 2)(n – 3)(n – 4)$ for $n > 1$ determines a sequence with the same first four terms, but whose fifth term is $a_5 = 10 + 2 + 6 = 18$, not 12.

For this reason, we are careful to say “find a rule…” rather than “find the rule…”

Different Definitions of Function

There are two routes for developing the function idea. The route we have followed is to define a function as an input-output rule, and to define the graph of the function as ordered pairs of input and output values.

Another more sophisticated route, which is common in school mathematics, is to define first a relation to be a set of ordered pairs, and then to define a function to be a relation with the “vertical line property.” A *function*, according to this definition, is what we have defined as the *graph of the function*.

The two routes lead to essentially the same class of objects. However, we have chosen the input-output definition because it is the definition used in college mathematics, particularly calculus. Further, the input-output definition is better adapted to understanding algebraic operations on functions and the operation of composition of functions.

MATH BACKGROUND

Growing the Formal Definition of Slope

Two beginning definitions of the slope of a line, the increase in the dependent variable (*y*) per unit increase in the independent variable (*x*), are given by:

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$ as we move from one point to another on the same line, and

$$\text{slope} = \frac{\text{difference in } y \text{ coordinates}}{\text{difference in } x \text{ coordinates}}$$ as we move from one point to another on the same line.

These informal definitions use language that accurately captures the essence of traditional definitions used in more advanced mathematics courses, such as:

For two points (x_1, y_1) and (x_2, y_2) the slope of the line segment joining them is $\frac{y_2 - y_1}{x_2 - x_1}$.

Slogan definitions are also popular, but have limitations because they are not complete. For example, the slogan "rise over run" $\left(\frac{\text{rise}}{\text{run}}\right)$ may be misleading because rise implies "up." The slogan "delta *x* over delta *y*" $\left(\frac{\Delta y}{\Delta x}\right)$ is only useful if one understands that Δ means "change in."

Our modest definition of slope grows gradually to the foundations of differential calculus, where students study rates of change of many nonlinear functions.

Why Can't You Divide by Zero?

Why can't you divide by zero? The mathematical reason is that the definition of "division by b" is multiplication by the multiplicative inverse (reciprocal) of b. The number zero has no multiplicative inverse — there is no number *a* satisfying $0 \bullet a = 1$, because $0 \bullet a = 0$ for all a. Consequently, we cannot divide by zero. Here are two additional ways to explain why you cannot divide by zero.

Explanation 1

$\frac{6}{0} = a$ implies that

$6 = a \bullet 0$

Therefore,

$6 = 0$,

which is absurd.

Explanation 2

Consider whole number division as repeated subtraction:

For 6 ÷ 2 (remove 3 groups of 2):

$$\begin{array}{r} 2\overline{)\ 6} \\ -2 \\ \hline 4 \\ -2 \\ \hline 2 \\ -2 \\ \hline 0 \end{array}$$

So, $\frac{6}{2} = 3$

For 6 ÷ 0 (remove ? groups of 0):

$$\begin{array}{r} 0\overline{)\ 6} \\ -0 \\ \hline 6 \\ -0 \\ \hline 6 \\ -0 \\ \hline 6 \end{array}$$

So, $\frac{6}{0} = ?$

And so on…

MATH BACKGROUND

Numerical and Categorical Data Sets

A data set is a collection of pieces of information about a population, often numbers, obtained from observation, questioning, or measuring. Data consisting of numbers is referred to as numerical data. Examples are height, weight, and age of students at a school.

A categorical data set is a data set consisting of attributes of a population. Some examples of categorical data sets for the population of students in a school are: whether a student has a pet, the birthday of each student, and a student's favorite course. Categorical data sets typically have only a finite number of categories. For instance, "Do you own pets?" has two categories (yes or no), and "What is your birthday?" has 366 possible categories.

A measurement data set is simply a numerical data set obtained from measuring something. Examples of measurement data sets are heights, weights, lengths, areas, volumes, and temperatures. Often the theoretically possible measurement values belong to an interval or other infinite subset of the number line. However, the measurements are rounded, effectively grouping the measurement values into classes.

Data can also consist of "attributes," such as blood type of students at a school, what language students speak at home, or the musical instruments students play. It is common for attributes to be labeled by numbers. For instance, the age of a student at a school can be viewed as an attribute.

Some numerical data might be treated as categorical data. For instance, the age of each student in the school can be viewed as a categorical data set, whose categories are represented by the integers from 5 through 18.

A common procedure in statistics is to group the data points of a data set into a finite number of classes, and to treat the classes as a categorical data set. This makes it possible to construct bar graphs or to otherwise understand better the data. For instance, students' T.V. viewing habits might be grouped into three classes: 2 hours or less per week, more than 2 hours but less than 10 hours per week, or 10 or more hours per week. Assigning each student to one of these then becomes a categorical data set with three categories.

Association and Causation

One of the goals of statistics is to determine whether two variables are related and, if so, to determine the strength of the relationship. If there is evidence of a relationship, we say the variables are associated, or there is an association between the variables. In the case of two numerical variables, we can look for an association by creating a "scatter plot," that is, by graphing data pairs in a coordinate plane. If the data points cluster on a curve, there is likely an association between the variables, and the equation of the curve suggests the functional relationship between the variables.

If the data points cluster along a straight line, we say the data suggests a linear association between the variables. There are various statistical tests for determining how near the data points are to lying on a straight line; that is, the strength of a linear association between the data points. The "correlation coefficient" for bivariate numerical data is one such measure, which students will study in a future statistics course. There are also various techniques for determining the line which fits the data best in some sense.

The fact that there is an association between two variables does not mean that a change in either variable can be ascribed to a change in the other. Other factors may cause changes in both variables simultaneously. The fact that there is a positive association between smoking and lung cancer does not by itself imply that smoking causes lung cancer. Further statistical and medical evidence is required to make that case. By the same token, the fact that there is a positive association between atmospheric carbon and global warming does not per se imply that increased carbon in the atmosphere causes global warming. Further statistical and physical evidence is required to establish causality.

Lines of Best Fit

In their first attempt to analyze a scatter plot, students estimate a line of best fit by inspection. Statisticians have a toolkit of tests and techniques for creating models to fit data. Some of these include linear regression (for data that roughly fits a linear function), quadratic regression (for data that roughly fits a quadratic function such as the height of a projectile), exponential regression (for data that roughly fits an exponential function such as exponential decay of a radioactive isotope), and multiple regression analysis (finding regression equations involving more than two variables). Students will study more formal ways to fit models to data in future statistics courses.

Linear regression is concerned with the prediction of one variable on the basis of measurements provided by another variable by means of a linear relationship. The term "regression" stems from the seminal nineteenth-century study by Francis Galton, who showed that the heights of offspring tend to be linearly related to the heights of parents, but that the heights of offspring tend to regress to the mean. (Tall parents tend to have tall children, but the children are closer to average in height than the parents.)

The line of best fit reported by hand calculators is the line determined by the "method of least squares." This method, developed independently by Gauss and Legendre in the early 1800's, was used by Gauss to accurately predict the path of the newly discovered asteroid Ceres in 1801. The method calls for finding an expression for the vertical deviation between each data point and the line (as a difference of y-coordinates), then choosing the slope and y-intercept of the line to minimize the sum of the squares of these deviations.

Categorical Variables and Frequencies

A categorical data set based on a population is sometimes referred to as a "categorical variable." Technically, a categorical variable is a function that assigns to each member of the population a category. Since a categorical data set has only a finite number of categories, a categorical variable can assume only a finite number of values, namely, the finitely-many possible categories.

Example of a population: students in a class.

Example of a categorical variable based on this population: a function that assigns pet type to each student (assigning something like fish, hamster, bird, no pet, etc. to each student).

The frequency of a category is the number of times that that the variable assumes that category as a value, that is, the number of members of the population belonging to that category.

Example for frequencies based on the categories above: 3 students have fish, 4 have a hamster, 1 has a bird, 7 have no pets, etc.

Using Data to Understand Our World

A key to understanding the world we live in is to collect data: temperature, wind velocity, rainfall, water levels, animal populations, cases of malaria, and so on. The data have a story to tell. It is the job of mathematics and statistics to read the story. We need mathematics and statistics to tell us what the trends are and with what certainty. Once we've read the story, it is our job to take steps to make the world a better place to live in.

Bivariate Data and Two-Way Tables

Bivariate data refers to two data sets based on the same population. For instance, the underlying population might be the students in the school, and the bivariate data set might report participation in performing arts and school sports teams.

For such a bivariate data set with two categorical variables, we are interested only in the distribution of values of the underlying variables. To display these in a frequency table, we use a two-way frequency table (a rudimentary kind of "contingency table") in which the rows correspond to one variable and the columns to the other variable. We label the rows with the categories of one variable, and we label the columns with the categories of the other variable. The frequencies are entered in the cell corresponding to the pair of categories. For clarity, we add a row of column totals on the bottom, and a column of row totals on the right.

As an example, Table I is a two-way frequency table for the bivariate data set about participation in performing arts and sports teams for students in a classroom. The rows correspond to the performing arts participation, and the columns correspond to sports participation. Each cell contains the number (frequency) of students and their responses.

Table I: Frequencies

	Sport	**No sport**	**Total**
Performing arts	8	2	10
No performing arts	16	14	30
Total	24	16	40

In order to understand the data better, we may convert the raw frequencies in Table I to percentages, and create a relative frequency table as in Table II. We are more familiar with the percentage scale running from 0 to 100 than a scale based on some number of students. It is easier to grasp the phrase "40% of the students" than the phrase "16 out of 40 students." though we should also like to be told the total number of students.

Table II: Relative Frequencies (n = 40)

	Sport	**No sport**	**Total**
Performing arts	20%	5%	25%
No performing arts	40%	35%	75%
Total	60%	40%	100%

There is another kind of relative frequency table that makes it easier to compare the frequency distributions of two rows (or two columns) with each other. For the table above, instead of using the percentages for the entire population, we use the percentages in each row of the number of students in the category corresponding to that row. In other words, we scale each row so that the total percentage for each row is 100%. If the two variables are independent, one expects the percentages in the rows to have the same distribution. There are various statistical tests that compare rows to determine if the rows are different enough to indicate that there is an association or contingency between the two variables.

For example, if we scale the relative frequencies in each row of Table II, we obtain a relative frequency table in which the relative frequencies are calculated row by row. Table III makes it easier to compare the distribution of sports participation based on participation in the performing arts. From this table, it is clear that those who participate in performing arts are also more likely to play a sport.

Table III: Relative Frequencies by Row

	Sport	**No sport**	**Total**
Performing arts (*n* = 10)	80%	20%	100%
No performing arts (*n* = 30)	53%	47%	100%

A similar table could be made by columns to compare the distribution of sports participation in performing arts based on participation in sports.

MATH BACKGROUND

Conjecture Versus Proof: Number Tricks

Four terms associated with making logical arguments are:

Inductive reasoning is a form of reasoning where the conclusion is supported by empirical evidence (such as examples), but is not proved.

Deductive reasoning is a form of reasoning in which the conclusion is justified by an argument based on definitions, known facts, and accepted rules of logic.

A conjecture is a statement that is proposed to be true, but has neither been proven to be true nor to be false.

Generalization is the process of formulating general concepts by abstracting common properties from specific cases.

For a "number trick," we see these various forms of arguments illustrated.

Step	A number trick	Using specific numbers		Using a variable
1	Choose a number	3	0.5	n
2	Multiply by 4	12	2	$4n$
3	Add 6	18	8	$4n + 6$
4	Subtract the original number	15	7.5	$3n + 6$
5	Divide by 3	5	2.5	$n + 2$
6	Subtract 2	3	0.5	n

Based on the results with specific numbers, we use inductive reasoning to conjecture that the result of applying these operations to any number is the number itself.

By following the steps of this trick using n to represent any number, we prove (using deductive reasoning) that the result is the original number for all numbers.

When students in a classroom select different numbers to test a number trick such as this one, they may convince themselves that the trick will always work. Their generalization is a conjecture because it has not been proven to be true nor shown to be false. The sort of question, "Will this trick work for all numbers?" is a very important one in mathematics. Certainly, it is impossible to try all numbers. The use of symbolic algebra for the purpose of generalization is an efficient way to prove this conjecture, and it provides a convincing way to show the usefulness of algebra.

MATH BACKGROUND

Euclidean Geometry is not the only Geometry

A geometry for a surface is a structure that allows us to measure distances and angles on the surface. It tells us what curves are the shortest distances between points. The three two-dimensional surfaces we may be most familiar with are (1) the plane (think of a piece of paper on a flat desktop, extended infinitely far in all directions), (2) the sphere (think of the surface of the earth), and (3) the disk (think of a blank CD disc, or a phonograph record without the grooves). Each of these surfaces has a special geometry of interest. For the plane, it is the Euclidean geometry, in which the shortest distance between two points is the straight-line segment joining them. In the plane, the sum of the angle measures of a triangle is 180 degrees (a "straight angle"). For the sphere, the geometry of most interest is spherical geometry, in which the shortest distance between two points is the arc of the great circle through the points. On the sphere, the sum of the angles of a triangle is always greater than 180 degrees. The geometry of a disk of interest to mathematicians is hyperbolic geometry, in which the shortest distance between two points is along the arc of the circle through the two points that is perpendicular to the edge of the disk. In hyperbolic geometry, the sum of the degree measures of the angles of a triangle is always less than 180 degrees.

We use the phrase "Euclidean geometry" in recognition of the Greek mathematician Euclid, who lived around 300 BCE. Euclid wrote a textbook on geometry in which the geometric properties of the plane were developed from certain "primitive notions" such as "points" and "lines," certain definitions such as for "triangle" and "parallel lines," and certain axioms such as the "parallel line postulate." Using these as starting points, Euclid derived other properties of the plane through a process of logical reasoning. Euclid's *Elements* was the main textbook used by geometry students for several centuries after the Renaissance, into the 1900's. It was important not only for the teaching of geometry, but also for training students to develop their own reasoning skills. These skills of deduction and logical reasoning are important not only for mathematicians and scientists but also in many professions such as law.

There are higher-dimensional geometries that are relevant to the universe we live in, such as the four-dimensional "Minkowski space" that incorporates space and time.

Informal Mathematical Vocabulary

As students advance through the grade levels, the everyday expressions they have learned for mathematical objects and ideas are gradually replaced by the more formal vocabulary of mathematics. Even though more precise mathematical words have been introduced, it is often still useful to have at hand the informal language for referring to ideas and for creating a picture in the mind of the student (or adult). Some examples of informal math language and their formal counterparts:

slide	translation
turn	rotation
flip	reflection
patty paper move	rigid motion
cover a figure exactly	be congruent to
input-output rule	function

It is very important to teach precision in mathematical reasoning and in mathematics vocabulary. At the same time, the strategic use of informal language can be an effective tool in making concepts more understandable and more memorable.

Basic Properties Shared by Translations, Rotations, and Reflections

Translations, rotations, and reflections have a number of properties in common:

1. **Translations, rotations, and reflections preserve distances.** In other words, if P and Q are two points in the plane, and if the transformation maps $P \rightarrow P'$ and $Q \rightarrow Q'$, then the distance from P to Q is equal to the distance from P' to Q'. Transformations that preserve distances are referred to as <u>isometries</u> or <u>rigid motions</u>. The rigid motions of the plane are exactly those transformations of the plane that are the result of successive applications of translations, rotations, and reflections. In fact, in school mathematics it is customary to take the definition of a <u>rigid motion of the plane</u> to be a sequence of translations, rotations, and reflections.

2. **Translations, rotations, and reflections map lines to lines.** More generally, any rigid motion takes lines to lines. Further, rigid motions take line segments to line segments and they take circles to circles.

3. **Translations, rotations, and reflections preserve parallelism.** Indeed, if two lines do not meet, then their images under any rigid motion cannot meet.

4. **Translations, rotations, and reflections preserve angle measure.** One way to see this is to use the SSS criterion of Euclidean geometry. Consider an angle $\angle QPR$ at a point P, determined by two line segments $\overline{PQ}$ and $\overline{QR}$ meeting at P. A rigid motion mapping $P \rightarrow P'$, $Q \rightarrow Q'$, and $R \rightarrow R'$ maps the triangle $\triangle PQR$ to a triangle $\triangle P'Q'R'$, whose sides must have the same lengths as the corresponding sides of $\triangle PQR$. By the SSS criterion, $\triangle PQR$ is congruent to $\triangle P'Q'R'$. Consequently, corresponding angles are congruent, and the angle at P has the same measure as the angle at P'.

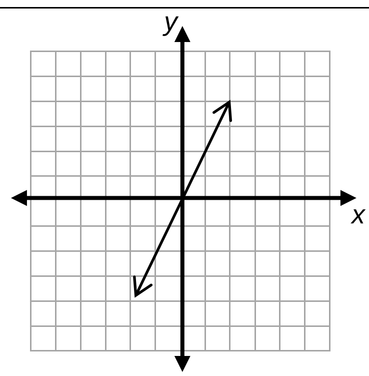
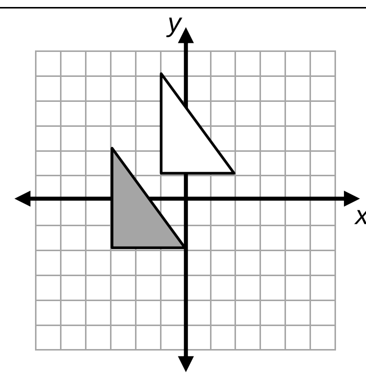
Middle school students are expected to learn these properties, to experiment with these properties, and to make observations that confirm the validity of these properties. However, formal verification of these properties is not a part of the Common Core State Standards for eighth grade mathematics.

Functions in Mathematics

The function concept is one of the most important and widespread concepts in mathematics. When the idea comes up in a specific field, the vocabulary is often adapted to give the flavor of the field. In probability and statistics, the functions of importance are defined on a sample space, and they are referred to as “random variables.” In functional analysis, the functions from one vector space to another are referred to as “operators” or “transformations.” In geometry, a function from one space to another is often referred to as a “transformation” or as a “map,” and we talk about “mapping” one space to another.

How we visualize functions graphically also varies from field to field. In probability and statistics, what is important is not the specific random variable but rather its “distribution” function, which describes how the values of the random variable are distributed probabilistically. Everyone is familiar with the bell-shaped curve of the normal distribution function, which represents the distribution of values of a normal random variable. In school algebra, we may visualize a function by representing its graph in a coordinate plane and graphing the output value (y-value) against the input value (x-value). In geometry, we may visualize functions by drawing arrows from points in the domain space to their image points in the range space. We may also visualize the function by sketching how the function transforms certain figures. This helps us to understand whether the function is a translation, rotation, or reflection, or whether the function is some sort of stretching or other operation.

The differences in describing and visualizing the functions of school algebra and geometry are portrayed in the table below, contrasting a typical algebra function on the line and a typical transformation of the plane.

verbal description	**a rule that assigns each real number to twice its value**	**a rule that shifts all points in the plane 2 units to the right and 3 units up**
symbolic notation	x maps to $2x$ $f(x) = 2x$ $x \to 2x$	(x, y) maps to $(x + 2, y + 3)$ $T(x, y) = (x + 2, y + 3)$ $(x, y) \to (x + 2, y + 3)$
graph		
graph interpretation	The x-coordinates represent the inputs and the y-coordinates represent the outputs. The set of input-output pairs is represented by the line.	A figure (shaded triangle) and its image (unshaded triangle) illustrate what is happening to points in the plane.

The Common Core State Standards postpone the introduction of function notations such as $f(x)$ and $T(x,y)$ to the high school curriculum. Consequently, the arrow notation is often used to describe transformations.

Approaching Congruence and Similarity through Transformational Geometry

One of the most transparent content changes in the 2010 Common Core State Standards in Mathematics is the introduction of transformational geometry in 8^{th} grade, leading towards a transformational approach to geometry in high school. Whereas classical synthetic geometry focuses on geometric constructions as in Euclid's *Elements*, transformational geometry focuses on certain classes of geometric transformations and the properties of figures that remain invariant under them.

The 8th grade standards focus on two classes of transformations of the plane (functions from the plane to the plane).

The first important class of transformations of the plane is the rigid motions, or isometries, which are the transformations that preserve distance (this unit). Any rigid motion can be represented as a sequence of translations, rotations, and reflections. Rigid motions map lines to lines, and they preserve lengths, areas, and measures of angles. In transformational geometry, two figures are defined to be congruent if one can be mapped onto the other by a sequence of rigid motions.

The second important class of transformations of the plane is the similarity transformations, which are compositions of rigid motions and dilations (Unit 10). Similarity transformations map lines to lines, and they preserve angles. However, in general they do not preserve lengths and areas. In transformational geometry, two figures are defined to be similar if one can be mapped onto the other by a sequence of similarity transformations.

The sequence of concepts studied that relate to geometry of the plane in 8^{th} grade is anchored in the core idea of function and leads to the properties of congruence and similarity. It can be described as follows:

functions → transformations of the plane ↗ rigid motions (translations, rotations, reflections) → congruent figures
↘ rigid motions and dilations (enlargements, reductions) → similar figures

The use of transformations as the foundation of geometry stems from the "Erlanger Programm" of Felix Klein. He laid the program out in an address he gave in 1872 upon becoming a professor of mathematics at the University of Erlangen in Germany. At the time, he was 23 years old. He eventually moved to Göttingen and helped to make it one of the most important centers for mathematics of his day. Around 1900, Klein became involved with math education; among other things pushing for introduction of the function concept and calculus ideas earlier in the German secondary math curriculum. Klein had an important effect on mathematics and math education in the United States, through the Americans who went to Germany to study at Göttingen, and through his students who took positions in the United States.

MATH BACKGROUND

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The first important class of transformations of the plane is the rigid motions, or isometries, which are the transformations that preserve distance (Unit 9). Any rigid motion can be represented as a sequence of translations, rotations, and reflections. Rigid motions map lines to lines, and they preserve lengths, areas, and measures of angles. In transformational geometry, two figures are defined to be congruent if one can be mapped onto the other by a sequence of rigid motions.

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Properties of Dilations

Dilations share many, but not all, of the properties of rigid motions. Some of the most important properties of dilations are:

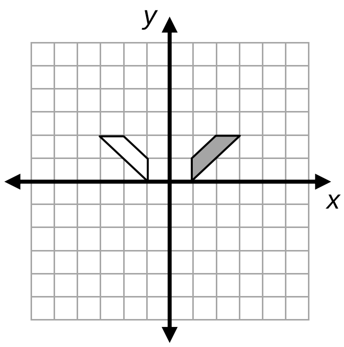
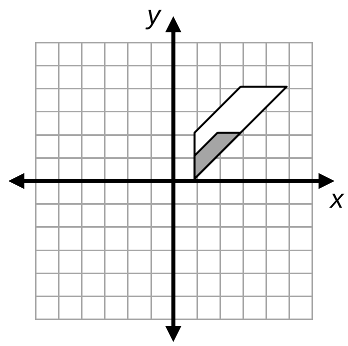
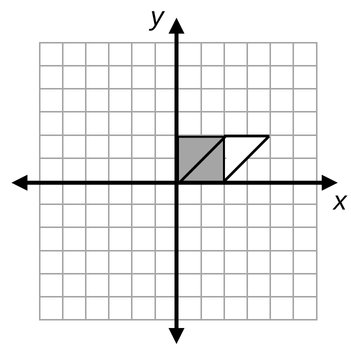
1. **Dilations map lines to lines.** The way to see this is to introduce a coordinate system with its origin at the center of the dilation. With respect to coordinates, the dilation then takes the form $(x, y) \longrightarrow (sx, sy)$, where $s > 0$ is the scale factor of the dilation. We calculate that the image of the line with the equation $y = mx + b$ is the line with the equation $y = mx + sb$. Thus, the image of a line under a dilation is a line with the same slope.

2. **Dilations preserve parallelism.** Indeed, if two lines do not meet, then their images under a dilation do not meet.

3. **Dilations preserve angle measure.** Consider an angle at a point P, determined by two lines intersecting at P. The dilation maps each of these lines to a parallel line through the image Q of P. Since the lines through Q are parallel to the lines through P, the angles they determine at Q have the same measure as the corresponding angles of the lines through P, and by checking which angle corresponds with which, one concludes that the angles at P have the same measures as their image angles at Q.

4. **In general, dilations do not preserve distances.** The only dilation (with scale factor $s > 0$) that preserves distances is the dilation with scale factor $s = 1$, that is, the identity transformation $(x, y) \longrightarrow (x, y)$.

Two examples and one non-example:

<table>
<tr><td>$(x, y) \longrightarrow (-x, y)$</td><td>$(x, y) \longrightarrow (2x, 2y)$</td><td>$(x, y) \longrightarrow (x + y, y)$</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Rigid motion (reflection) preserves lines, angle measures, and distances; and also lengths and areas.</td><td>Dilation preserves lines and angle measures, but not distances.</td><td>Shear preserves lines, but neither angle measures nor distances.</td></tr>
</table>

Congruence and Similarity are Equivalence Relations

The relation "*S* is congruent to *T*" is an equivalence relation on the family of subsets *S*, *T*, … of the plane.

Recall that an equivalence relation (~) on a set of objects has three properties:

Reflexive Property: $a \sim a$ for any a belonging to the set.

Symmetric Property: If $a \sim b$, then $b \sim a$.

Transitive Property: If $a \sim b$ and $b \sim c$, then $a \sim c$.

Three properties of rigid motions, which are easily verified, are needed in order to show that congruence is an equivalence relation. They are:

1. The identity transformation, $\mathrm{I}(P) = P$ for all *P*, is a rigid motion.
2. The inverse of a rigid motion is a rigid motion.
3. The composition (successive application) of rigid motions is a rigid motion.

Similarity transformations are transformations that are compositions of rigid motions and dilations. Again, it is easy to verify that:

1. The identity transformation, $\mathrm{I}(P) = P$ for all *P*, is a similarity transformation.
2. The inverse of a similarity transformation is a similarity transformation.
3. The composition of similarity transformations is a similarity transformation.

From these properties it follows that similarity is an equivalence relation.

The Angle-Angle (A-A) Criterion for Similarity of Triangles

The A-A criterion asserts that if two angles of one triangle are congruent respectively to two angles of another triangle, then the triangles are similar.
In fact, more can be said. If $\triangle ABC$ and $\triangle DEF$ are triangles for which the angles at the vertices *A* and *B* are congruent respectively to the angles at the corresponding vertices *D* and *E*, then there is a sequence of maps consisting of a translation, followed by a rotation, followed by a dilation, possibly followed by a reflection, whose composition maps $\triangle ABC$ onto $\triangle DEF$. First translate *A* to *D*, then rotate about *A* so that the side *AB* goes in the same direction as the side *DE*, then dilate so that the side *AB* lands exactly on the side *DE*, then if necessary reflect so that *C* lands on *F*. Each of these maps sends any triangle to a similar triangle, so $\triangle ABC$ is similar to $\triangle DEF$.

Note: The A-A-criterion is a theorem about triangles. It fails for quadrilaterals, and even for rectangles. For example, a rectangle has four right angles, but it is not necessarily similar to a square.

Similarity and Slope

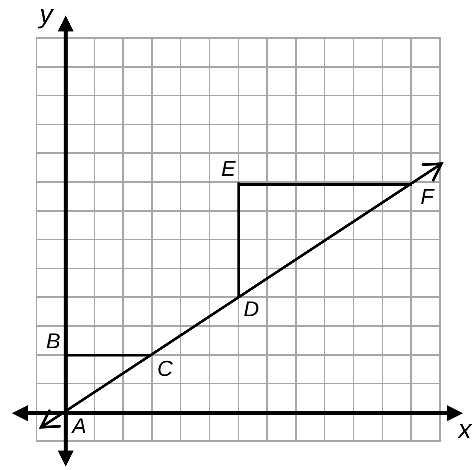
To find the slope of the line to the right, we take two points on the line, say A and C, and we calculate the rise over the run, which in this case yields

$$\text{slope} = \frac{|AB|}{|BC|}$$

What do we get if we use two other points, say D and F, to calculate the rise over the run?

In the figure at the right, $\triangle ABC$ and $\triangle DEF$ are two right triangles.

The line through B and C is parallel to the line through E and F, since both are horizontal, and our line is a transversal to these parallel lines.



Consequently, the corresponding angles at C of $\triangle ABC$ and at F of $\triangle DEF$ are congruent. By the A-A criterion for similarity of triangles, the triangles $\triangle ABC$ and $\triangle DEF$ are similar. Consequently, the sides of the triangles are proportional:

$$\frac{|AB|}{|BC|} = \frac{|DE|}{|EF|}$$

In other words, we obtain the same slope for the line no matter which pair of points we use on the line to calculate the slope.