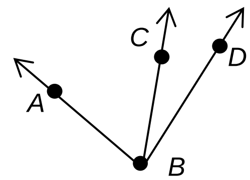
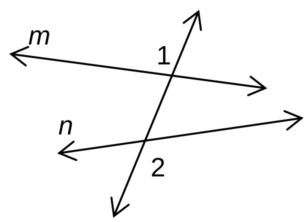
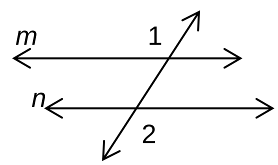
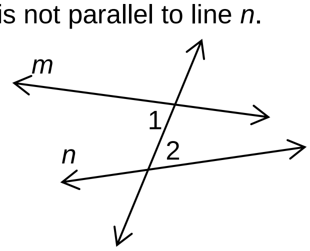
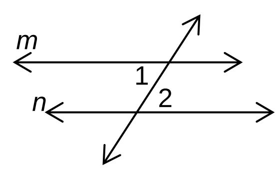
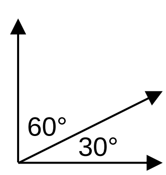
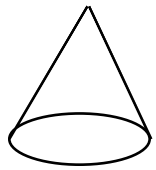
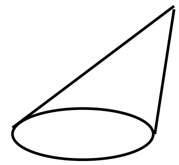
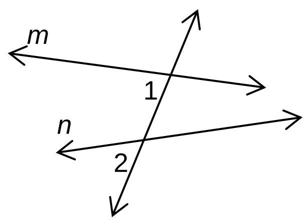
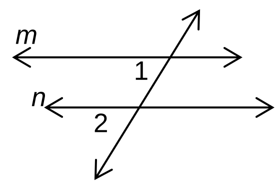
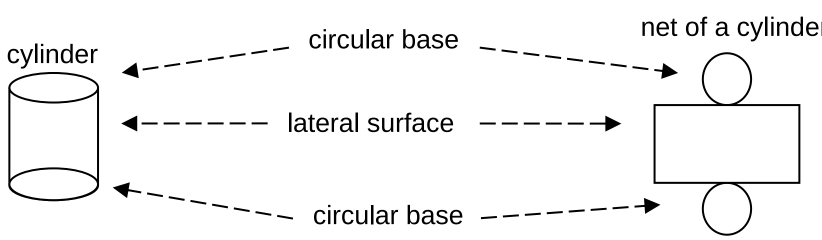
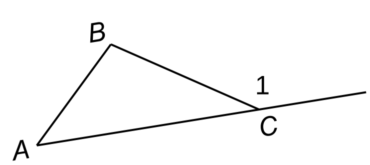
TEACHER EDITION

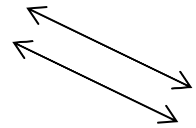
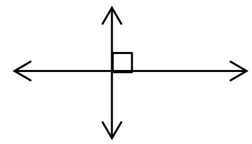
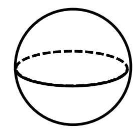
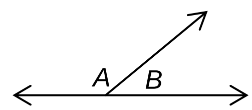
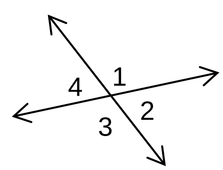
MathLinks GRADE 8

COMPILED STUDENT RESOURCES

STUDENT RESOURCES

Word or Phrase	Definition
adjacent angles	Two angles are <u>adjacent</u> if they have the same vertex and share a common ray, and they lie on opposite sides of the common ray. $\angle ABC$ and $\angle CBD$ are adjacent angles. 
alternate exterior angles	When two lines in a plane are cut by a transversal, two angles on opposite sides of the transversal and outside the two lines are referred to as <u>alternate exterior angles</u>. When parallel lines are cut by a transversal, alternate exterior angles have the same measure. Line *m* is not parallel to line *n*.  $\angle 1$ and $\angle 2$ are alternate exterior angles. Line *m* is parallel to line *n.*  $\angle 1$ and $\angle 2$ are alternate exterior angles. $\lvert\angle 1\rvert = \lvert\angle 2\rvert$
alternate interior angles	When two lines in a plane are cut by a transversal, two angles on opposite sides of the transversal and between the two lines are referred to as <u>alternate interior angles</u>. When parallel lines are cut by a transversal, alternate interior angles have the same measure. Line *m* is not parallel to line *n*.  $\angle 1$ and $\angle 2$ are alternate interior angles. Line *m* is parallel to line *n.*  $\angle 1$ and $\angle 2$ are alternate interior angles. $\lvert\angle 1\rvert = \lvert\angle 2\rvert$
complementary angles	Two angles are <u>complementary</u> if the sum of their measures is 90°. Two angles that measure 30° and 60° are complementary. 

Word or Phrase	Definition
cone	A circular <u>cone</u> is a figure in space consisting of a circle in a plane (called the <u>base</u> of the cone), a point off the plane (called the <u>vertex</u> of the cone), and all the straight line segments joining the vertex to the base. If the line joining the vertex of the cone to the center of its base is perpendicular to the base, the cone is a <u>right circular cone</u>. Otherwise it is an <u>oblique circular cone</u>.  right circular cone  oblique circular cone
corresponding angles	When two lines in a plane are cut by a transversal, two angles that appear on the same side of the transversal in the same relative location are referred to as <u>corresponding angles</u>. When parallel lines are cut by a transversal, corresponding angles have the same measure. Line *m* is not parallel to line *n*.  $\angle 1$ and $\angle 2$ are corresponding angles. Line *m* is parallel to line *n.*  $\angle 1$ and $\angle 2$ are corresponding angles. $\lvert\angle 1\rvert = \lvert\angle 2\rvert$
cylinder	A (right circular) <u>cylinder</u> is a figure in three-dimensional space that has two parallel circular bases. These circles are connected by a curved surface, called the <u>lateral surface</u>, which is a “rolled up” rectangle. Most soup cans have the shape of a right circular cylinder. 
exterior angle of a triangle	An <u>exterior angle of a triangle</u> is an angle formed by a side of the triangle and an extension of its adjacent side. $\angle 1$ is an exterior angle of ΔABC. 

Word or Phrase	Definition
parallel	Two lines in a plane are parallel if they do not meet. Two line segments in a plane are parallel if the lines they lie on are parallel. 
perpendicular	Two lines are perpendicular if they intersect at right angles. 
sphere	A sphere is a closed surface in three-dimensional space consisting of all points at a fixed distance (the radius) from a specified point (the center). 
supplementary angles	Two angles are supplementary if the sum of their measures is 180°. Any two right angles are supplementary, because the sum of their measures is 90° + 90° = 180°. Angles *A* and *B* are supplementary because they determine a straight line, or 180°. 
transversal	A transversal is a line that passes through two or more other lines.
vertical angles	Two angles are vertical angles if they are the opposite angles formed by a pair of intersecting lines. When two lines intersect at a point, they form two pairs of vertical angles with vertex at the point. $\angle 1$ and $\angle 3$ are vertical angles. $\angle 2$ and $\angle 4$ are vertical angles. 

Some Properties of Equality

Properties of equality govern the manipulation of equations (mathematical sentences).

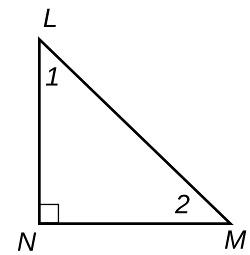
For any three numbers *a*, *b*, and *c*:

- ✓ Addition property of equality
  (Subtraction property of equality)
  If $a = b$ and $c = d$, then $a + c = b + d$.
- ✓ Multiplication property of equality
  (Division property of equality)
  If $a = b$ and $c = d$, then $ac = bd$
- ✓ Reflexive property of equality
  $a = a$
- ✓ Symmetric property of equality
  If $a = b$, then $b = a$
- ✓ Transitive property of equality
  (Substitution property)
  If $a = b$, and $b = c$, then $a = c$

Geometry Notation

Here are some geometry notations used in these lessons.

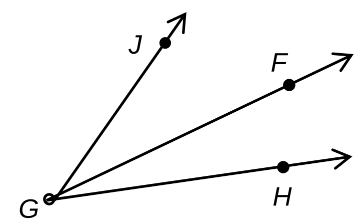
- Points are named by capital letters.
- The symbol for triangle is Δ.
- The symbol for angle is $\angle$.
- Absolute value signs are used to denote nonnegative quantities that measure the “size” of something, such as length or angle measure.



The measure of an angle called $\angle N$ is denoted by $|\angle N|$. The small square at N indicates that $\angle LNM$ is a right angle, that is, that $|\angle LNM| = 90°$.

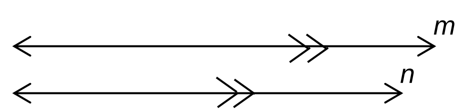
In naming a triangle, vertices may be listed in either a clockwise or counter - clockwise direction. For example, the triangle may be named ΔLMN or ΔLNM.

In naming an angle, vertices may be listed in either a clockwise or counterclockwise direction. In the triangle above, the angle at the top can be denoted by $\angle NLM$, $\angle MLN$, $\angle L$ or $\angle 1$.



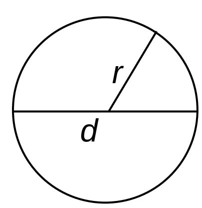
The pair of adjacent angles to the right are $\angle FGJ$ and $\angle HGF$. Using $\angle G$ to name an angle is unclear They share the common ray $\overrightarrow{GF}$. The two adjacent angles together form the angle $\angle JGH$.

The arrows on the lines m and n indicate that they are parallel.



Formulas for Circles

Let r = radius of a circle.
Let d = diameter of a circle.



Circumference: $C = \pi d$ or $C = 2\pi r$

Area: $A = \pi r^2$

Volume Formulas

Here are some volume formulas from this unit.

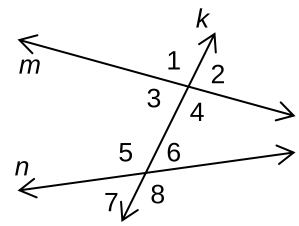
Volume of a Rectangular Prism	Volume of a Cylinder
Let ℓ = length and w = width of rectangular base. $V = Bh$ Area of base (B) = ℓw Therefore, $V = \ell wh$	Let r = radius of the circular base. $V = Bh$ Area of base (B) = πr^2 Therefore, $V = \pi r^2 h$

Volume of a Cone	Volume of a Sphere
Through experimentation, observe that the volume of a cone is $\frac{1}{3}$ of the volume of a cylinder with the same height and base. Let r = radius of the circular base $V = \frac{1}{3}Bh$ Area of base (B) = πr^2 Therefore, $V = \frac{1}{3}\pi r^2 h$	Through experimentation, observe that the volume of a sphere is $\frac{2}{3}$ of the volume of a cylinder whose diameter and height are the same as the diameter of the sphere. Use substitution to derive the formula of a sphere. Let r = radius of the sphere and cylinder Then height (h) of cylinder = $2r$ Volume of cylinder = $\pi r^2 (2r) = 2\pi r^3$ Observe that volume of sphere is $\frac{2}{3}$ of the volume of a cylinder. Therefore, $V_{sphere} = \frac{2}{3} \bullet 2\pi r^3 = \frac{4}{3}\pi r^3$

Transversals and Parallel Lines

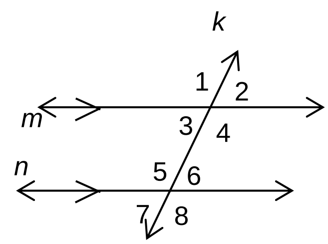
In this figure, line *k* is a transversal. Lines *m* and *n* are NOT parallel.

When two lines in a plane are cut (crossed) at two points by a transversal, eight angles are created. Some of these pairs of angles have special names.



corresponding angles		alternate interior angles	alternate exterior angles
$\angle 1$ and $\angle 5$	$\angle 2$ and $\angle 6$	$\angle 3$ and $\angle 6$	$\angle 1$ and $\angle 8$
$\angle 3$ and $\angle 7$	$\angle 4$ and $\angle 8$	$\angle 4$ and $\angle 5$	$\angle 2$ and $\angle 7$

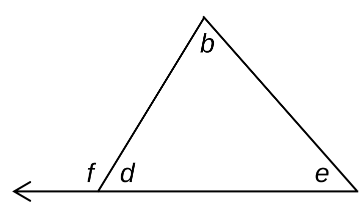
Here are three important properties of the angles formed when a transversal cuts two parallel lines.



1. If two parallel lines are cut by a transversal, then alternate interior angles have the same measure.
   Example: $|\angle 3| = |\angle 6|$ and $|\angle 4| = |\angle 5|$
2. If two parallel lines are cut by a transversal, then alternate exterior angles have the same measure.
   Example: $|\angle 1| = |\angle 8|$ and $|\angle 2| = |\angle 7|$
3. If two parallel lines are cut by a transversal, then corresponding angles have the same measure.
   Example: $|\angle 2| = |\angle 6|$ and $|\angle 4| = |\angle 8|$

Interior and Exterior Angles in Triangles

Here are two important facts about angle sums in triangles.



1. The sum of the measures of the angles in a triangle is equal to 180°.

   $|\angle d| + |\angle b| + |\angle e| = 180°$

2. The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent interior angles.

   $|\angle b| + |\angle e| = |\angle f|$

STUDENT RESOURCES

Word or Phrase	Definition
converse of the Pythagorean theorem	The converse of the Pythagorean theorem states that if the sum of the squares of the lengths of two sides of a triangle is equal to the square of the length of the third side, then the triangle is a right triangle. See Pythagorean theorem. If the lengths of the sides of a triangle are 3, 4, and 5 units respectively, then the triangle is a right triangle, because $3^2 + 4^2 = 5^2$.
hypotenuse	The hypotenuse of a right triangle is the side of the triangle opposite the right angle. It is the longest side in a right triangle.
integers	The integers are the whole numbers and their opposites. They are the numbers 0, 1, 2, 3, … and -1, -2, -3, … .
irrational numbers	Irrational numbers are real numbers whose decimal expansions continue infinitely without continuously repeating the same block of digits. The irrational numbers are the real numbers that are not rational. $\sqrt{2}$, , and 0.101001000100001... are irrational numbers and cannot be written as quotients of integers.
legs	The legs of a right triangle are the two sides of the triangle adjacent to the right angle.
natural numbers	The natural numbers are the numbers 1, 2, 3, … .Natural numbers are also referred to as counting numbers.
perfect square	A perfect square, or square number, is a number that is the square of a natural number. The area of a square with a natural number side-length is a perfect square. The perfect squares are $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, … .
Pythagorean theorem	The Pythagorean theorem states that the sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse. See converse of the Pythagorean theorem. $a^2 + b^2 = c^2$ If the lengths of the legs of a right triangle are 5 and 12 units respectively, then the hypotenuse has length 13 units, because $13^2 = 5^2 + 12^2$.
radical expression	A radical expression is an expression involving a root, such as a square root. In a radical expression, the symbol $\sqrt{}$ is called a radical sign, and the number under the radical sign is called the radicand. $\sqrt{5}$ is a radical expression. The radicand is 5.

Word or Phrase	Definition
rational numbers	Rational numbers are quotients of integers. Rational numbers can be expressed as $\frac{m}{n}$, where m and n are integers and $n \neq 0$. $\frac{3}{5}$ is rational because it *is* a quotient of integers. 4, $2\frac{1}{3}$, 0.7, and $0.\overline{25}$ are rational numbers because they *can be* expressed as quotients of integers ($4 = \frac{4}{1}$; $2\frac{1}{3} = \frac{7}{3}$; $0.7 = \frac{7}{10}$; $0.\overline{25} = 0.2525.... = \frac{25}{99}$). $\sqrt{2}$ and π are NOT rational numbers. They *cannot be* expressed as a quotient of integers.
real numbers	Real numbers refer to the rational numbers and irrational numbers together. Each real number has a decimal name (address) locating it on the real number line.
repeating decimal	A repeating decimal is a decimal that ends in repetitions of the same block of digits. A "repeat bar" can be placed above the digits that repeat. A terminating decimal is regarded as a repeating decimal that ends in all zeros. Repeating decimals represent rational numbers. $\frac{2}{9} = 0.22222... = 0.\overline{2}$ $\frac{2}{11} = 0.181818... = 0.\overline{18}$ these repeating decimals do NOT terminate $\frac{1}{2} = 0.50000... = 0.5\overline{0} = 0.5$ $\frac{3}{4} = 0.750000... = 0.75\overline{0} = 0.75$ these repeating decimals do terminate
square of a number	The square of a number is the product of the number with itself. The square of 5 is 25, since $5^2 = (5)(5) = 25$. The square of -5 is also 25, since $(-5)^2 = (-5)(-5) = 25$. This is different than $-5^2 = -(5)(5) = -25$.
square root	A square root of a number n is a number whose square is equal to n, that is, a solution of the equation $x^2 = n$. The positive square root of a number n, written $\sqrt{n}$, is the positive number whose square is n. Except where otherwise noted, the term "the square root of *n*" refers to the positive square root. $\sqrt{25} = 5$, because $5^2 = (5)(5) = 25$
terminating decimal	A terminating decimal is a repeating decimal whose digits are eventually a repeating 0 from some point on. The final 0's in the expression for a terminating decimal are usually omitted. 4.6200000... = 4.62. It is a terminating decimal with value $4 + \frac{6}{10} + \frac{2}{100}$.
whole numbers	The whole numbers are the natural numbers together with 0. They are the numbers 0, 1, 2, 3, … .

Numbers Squared

Why do we say that a number raised to the second power is "squared"? The reason has to do with the area formula for squares. The area of a square of side length s is given by

$$\text{area} = s \bullet s = s^2.$$

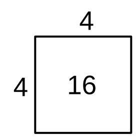
A square with side length 4 units has area "4 squared" = $4^2 = 16$ square units.

What about "square root" – where does that term come from?

Here the reason is that a "root" can also refer to the solution of an equation. A "square root" has to do with finding the side length of a square of a given area; that is, of solving the equation $s^2 = A$. For a given area A, the side length s of the square with area A is

side length = $s = \sqrt{A}$ = "square root of *A*."

A square with area 16 square units has side length $\sqrt{16}$ = 4 units.



→ $4^2 = 16$ and $\sqrt{16} = 4$

Square Roots: Estimates Versus Exact Value

Q: What is the square root of 9? *A*: We know that $3^2 = 9$, so $\sqrt{9} = 3$.

Q: What is the square root of 7? *A*: We know of no rational number that, when squared, is equal to 7.

Using the square root function on a simple calculator, we get an approximation to several decimal places: $\sqrt{27} \approx 5.196152$, and then by multiplication: $(5.196152)^2 = (5.196152)(5.196152) = 26.9999956$.

Find another approximation for $\sqrt{27}$ using a calculator with greater capacity, then square that number, and it will still not be exactly equal to 27.

We know that $\sqrt{27}$ is an irrational number and its decimal expansion is infinite with no block of digits that repeats.

So how do we write $\sqrt{27}$? The only way to write it exactly is to leave it in square root form.

If we choose to approximate $\sqrt{27}$, the simplest way may be to state which two consecutive integers it is between. We know that $5^2 = 25$ and $6^2 = 36$, and we also know that 27 is between 25 and 36, so $\sqrt{27}$ is between 5 and 6.

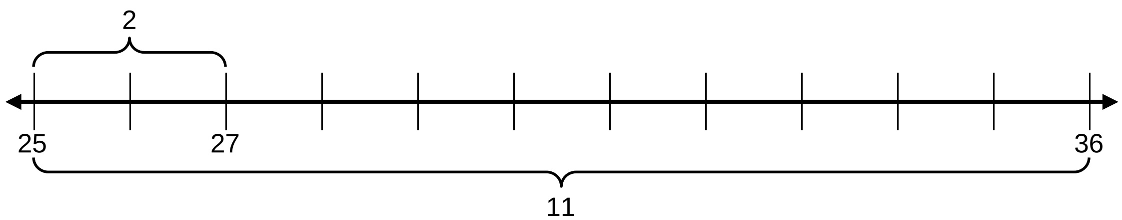
See the next box for an estimation method from lesson 2.1 that is more accurate.

Estimating Square Roots with Greater Accuracy: Linear Interpolation

The following two strategies may be helpful for square root estimation. Note: the method illustrated below is referred to as "linear interpolation."

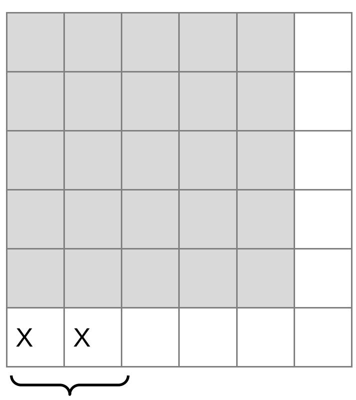
Find an estimate of $\sqrt{27}$ as a mixed number and as a whole number with decimal remainder.

Strategy 1: Since $\sqrt{27}$ is between $\sqrt{25}$ and $\sqrt{36}$ (25 and 36 are perfect squares), provide a "magnified" portion of a number line from 25 to 36.



The distance from 25 to 27 is $\frac{2}{11}$ of the distance from 25 to 36. Hence, the distance from $\sqrt{25}$ to $\sqrt{27}$ should be about $\frac{2}{11}$ of the distance from $\sqrt{25}$ to $\sqrt{36}$. In other words, the distance from 5 to $\sqrt{27}$ should be about $\frac{2}{11}$ of the distance from 5 to 6. Thus $\sqrt{27} \approx 5\frac{2}{11}$.

Strategy 2: Since 25 and 36 are perfect squares, on grid paper, draw a 5 x 5 square inside a 6 x 6 square as illustrated below.



The two X's represent the 26th and 27th unit squares.

- The larger square is $6 \times 6 = 6^2 = 36$ unit squares.
- The smaller, shaded square inside is $5 \times 5 = 5^2 = 25$ unit squares.

So, $\frac{27-25}{36-25} = \frac{2}{11}$

Therefore, $\sqrt{27}$ is about $5\frac{2}{11}$. (calculator check: $5\frac{2}{11} = 5.\overline{18}$ and $\sqrt{27} \approx 5.196$)

A Clever Procedure: Writing a Repeating Decimal as a Quotient of Integers

Any repeating decimal can be written as a quotient of integers. Therefore, all repeating decimals are rational. The following algebraic idea is used to change a repeating decimal to a quotient of integers.

Example 1: Change $0.1\overline{6} = 0.16666\ldots$

$10x = 1.66666...$ (2)

Let $\boldsymbol{x} = \mathbf{0.16666...}$ **(1)**

$9x = 1.5$ (3)

$x = \frac{1.5}{9} = \frac{15}{90} = \frac{1}{6}$ (4)

- Notice that step 2 is above step 1
- The “trick” is to multiply both sides of the equation in step 1 by a power of 10 that will “line-up” the repeating portion of the decimal.
- Subtract the expressions in step 1 from step 2. This results in a step 3 equation that has a terminating decimal. Solve for x in step 4 and simplify the result so it is a quotient of integers

Example 2: Change $0.\overline{7} = 0.77777\ldots$

$10x = 7.777777...$ (1)

Let $\boldsymbol{x} = \mathbf{0.7777777...}$ **(2)**

$9x = 7.000000...$ (3)

$x = \frac{7}{9}$ (4)

Ask yourself:

How many digits are repeating?
one

What do I multiply both sides by?
10 (a power of 10 with one zero)

Example 3: Change $0.\overline{45} = 0.454545\ldots$

$10x = 45.454545...$ (1)

Let $\boldsymbol{x} = \mathbf{0.45454545...}$ **(2)**

$99x = 45.00000...$ (3)

$x = \frac{45}{99} = \frac{15}{33} = \frac{5}{11}$ (4)

Ask yourself:

How many digits are repeating?
two

What do I multiply both sides by?
100 (a power of 10 with two zeros)

STUDENT RESOURCES

Word or Phrase	Definition
conjecture	A conjecture is a statement that is proposed to be true, but has not been proven to be true nor to be false. After creating a table of sums of odd numbers such as 1 + 3 = 4, 1 + 5 = 6, 5 + 7 = 12, 3 + 9 = 12, etc., we may make a conjecture that the sum of any two odd numbers is an even number. This conjecture can be proven to be true.
cube of a number	The cube of a number n is the number $n^3 = n \bullet n \bullet n$. The cube of -5 is $(-5)^3 = (-5)(-5)(-5) = -125$.
cube root	The cube root of a number n is the number whose cube is equal to n. That is, the cube root of n is the value of x such that $x^3 = n$. The cube root of n is written $\sqrt[3]{n}$. The cube root of -125 is $\sqrt[3]{-125} = -5$, because $(-5)^3 = (-5)(-5)(-5) = -125$.
exponent notation	The exponent notation b^n (read as " b to the power n ") is used to express n factors of b. The number b is the base, and the natural number n is the exponent. Exponent notation is extended to arbitrary integer exponents by setting $b^0 = 1$ and $b^{-n} = \frac{1}{b^n}$. $2^3 = 2 \bullet 2 \bullet 2 = 8$ (the base is 2 and the exponent is 3) $3^2 \bullet 5^3 = 3 \bullet 3 \bullet 5 \bullet 5 \bullet 5 = 1{,}125$ (the bases are 3 and 5) $2^0 = 1$ $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
radical expression	A radical expression is an expression involving a root, such as a square root. $\sqrt{20}$ and $5\sqrt{3}$ are radical expressions.
scientific notation	Scientific notation for a positive number represents the number as a product of a decimal between 1 and 10 and a power of 10. It is typically used to write either very large numbers or very small numbers. In scientific notation, the number 245,000 is written as 2.45×10^5. In scientific notation, the number 0.0063 is written as 6.3×10^{-3}.

Numbers Squared and Cubed

Why do we say that a number raised to the second power is "squared"? The reason has to do with the area formula for squares. The area of a square of side length s is given by

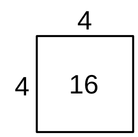
$$\text{area} = s \bullet s = s^2 .$$

A square with side length 4 units has area "4 squared" = 4^2 = 16 square units.

What about "square root" – where does that term come from?

Here the reason is that a "root" can also refer to the solution of an equation. A "square root" has to do with finding the side length of a square of a given area; that is, of solving the equation $s^2 = A$. For a given area A, the side length s of the square with area A is side length = $s = \sqrt{A}$ = "square root of A."

A square with area 16 square units has side length $\sqrt{16}$ = 4 units.



$\longrightarrow$ $4^2 = 16$ and $\sqrt{16} = 4$

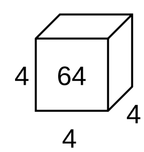
Why do we say that a number raised to the third power is "cubed"? In this case, the answer has to do with the volume formula for cubes. The volume of a cube with side length s is given by

$$\text{volume} = s \bullet s \bullet s = s^3 .$$

A cube with side length 4 units has volume "4 cubed" = 4^3 = 64 cubic units.

In turn, a "cube root" has to do with finding the side length of a cube of a given volume, that is, of solving the equation $s^3 = V$. For a given volume V, the side length s of the cube with volume V is side length = $s = \sqrt[3]{V}$ = "cube root of V."

A cube with volume 64 cubic units has side length $\sqrt[3]{64}$ = 4 units.



$\longrightarrow$ $4^3 = 64$ and $\sqrt[3]{64} = 4$

Although we assume here that V is positive, the cube root of a negative number can be found by solving the equation, $s^3 = V$. The square root of a negative number is not a real number.

Squaring a number and finding the square root of a number are inverse operations. Similarly, cubing a number and finding the cube root of a number are inverse operations.

Three Facts and Three Rules for Exponents		
Definitions and Rules		**Example**
Meaning of positive exponent:	$x^m = x \bullet x \bullet \ldots \bullet x$ (m factors)	$3^4 = 3 \bullet 3 \bullet 3 \bullet 3$ (4 factors of 3)
Fact about zero as an exponent:	$x^0 = 1,\ x \neq 0$	$3^0 = 1$, (0^0 is not defined)
Fact about a negative exponent:	$x^{-a} = \frac{1}{x^a},\ x \neq 0$	$3^{-2} = \frac{1}{3^2}$, (0 cannot be in the denominator because division by 0 is not defined)
Product rule for exponents:	$x^a \bullet x^b = x^{a+b}$	$3^2 \bullet 3^1 = 3 \bullet 3 \bullet 3 = 3^{2+1} = 3^3$
Power rule for exponents:	$(x^a)^b = x^{a \bullet b}$	$(3^2)^3 = 3^2 \bullet 3^2 \bullet 3^2 = 3^{3 \bullet 2} = 3^6$
Quotient rule for exponents:	$\frac{x^a}{x^b} = x^{a-b},\ x \neq 0$	$\frac{3^4}{3^6} = \frac{3 \bullet 3 \bullet 3 \bullet 3}{3 \bullet 3 \bullet 3 \bullet 3 \bullet 3 \bullet 3} = 3^{4-6} = 3^{-2}$, (0 cannot be in the denominator because division by 0 is not defined)
The three rules above apply to expressions with the same base numbers. For example: $5^3 \bullet 4^2 = (5 \bullet 5 \bullet 5) \bullet (4 \bullet 4)$, and the product rule does not apply.		

Making Sense of Zero and Negative Exponents			
These patterns show that the definitions for zero and negative exponents are reasonable.			
Pattern: Divide by 2	**Result of the division**	**Pattern as a product**	**Pattern in exponent form**
	Start with 8	$2 \bullet 2 \bullet 2$	2^3
$8 \div 2$	4	$2 \bullet 2$	2^2
$4 \div 2$	2	2	2^1
$2 \div 2$	1	1	2^0
$1 \div 2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2^1}$ or 2^{-1}
$\frac{1}{2} \div 2$	$\frac{1}{4}$	$\frac{1}{2 \bullet 2}$	$\frac{1}{2^2}$ or 2^{-2}
$\frac{1}{4} \div 2$	$\frac{1}{8}$	$\frac{1}{2 \bullet 2 \bullet 2}$	$\frac{1}{2^3}$ or 2^{-3}

Scientific Notation				
Given number	**Related decimal between 1 and 10**	**Power of 10**	**Number in scientific notation**	**Reasoning**
120,000,000	1.2	10^8	1.2×10^8	The given number is 10^8 times 1.2; adjust place values by multiplication.
0.0000345	3.45	10^{-5}	3.45×10^{-5}	3.45 is 10^5 times the given number; adjust place values by multiplication.

Some of the benefits of scientific notation:

(1) Scientific notation is useful for writing numbers with very large or very small values in a compact way.

(2) The power of 10 gives an immediate clue to the relative size of the number.

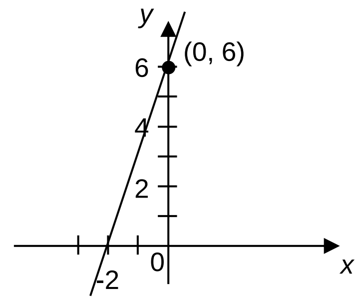
Error alert!

When comparing numbers in scientific notation such as 2.5×10^{12} and 8.76×10^8, a common mistake is to focus on the fact that 8.76 > 2.5. **Focus on the exponent!**

$$2.5 \times 10^{12} = 2{,}500{,}000{,}000{,}000$$
$$8.76 \times 10^{8} = 876{,}000{,}000$$

STUDENT RESOURCES

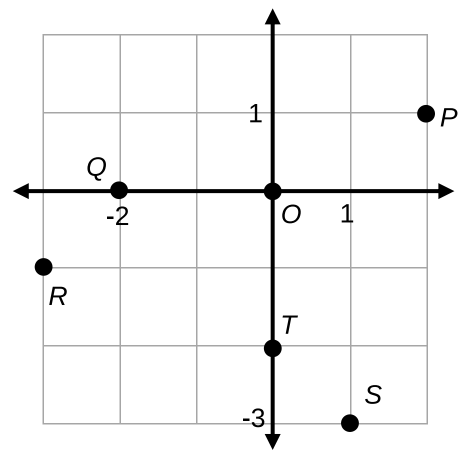
Word or Phrase	Definition
coefficient	A coefficient is a number or constant factor in a term of an algebraic expression. In the expression $3x + 5$, 3 is the coefficient of the term $3x$, and 5 is the constant term.
dependent variable	A dependent variable is a variable whose value is determined by the values of the independent variables. See independent variable.
function	A function is a rule that assigns to each input value exactly one output value. For $y = 3x + 6$, any input value, say $x = 10$, has a unique output value, in this case $y = 36$. For $y = x^2 + 1$, $x = 2$ has the unique output value $y = 2^2 + 1 = 5$.
graph of a function	The graph of a function is the set of all ordered pairs (x, y) where y is the output for the input value x. If x and y are real numbers, then we can represent the graph of a function as points in the coordinate plane.
independent variable	An independent variable is a variable whose value may be specified. Once specified, the values of the independent variables determine the values of the dependent variables. For the equation $y = 3x$, y is the dependent variable and x is the independent variable. We may assign a value to x. The value assigned to x determines the value of y.
input-output rule	An input-output rule for a sequence of values is a rule that establishes explicitly an output value for each given input value. **input value (x)**: 1, 2, 3, 4, 5, x **output value (y)**: 1.5, 3, 4.5, 6, 7.5, $1.5x$ In the table above, the input-output rule could be $y = 1.5x$. To get the output value, multiply the input value by 1.5. If $x = 100$, then $y = 1.5(100) = 150$.
proportional	Two variables are proportional if the values of one are the same constant multiple of the corresponding values of the other. The variables are said to be in a proportional relationship, and the constant is referred to as the constant of proportionality. If Wrigley eats 3 cups of kibble each day, then the number of cups of kibble is proportional to the number of days. If x is the number of days, and y is the number of cups of kibble, then $y = 3x$. The constant of proportionality is 3.
unit rate	The unit rate associated with a ratio $a : b$ of two quantities a and b, $b \neq 0$, is the number $\frac{a}{b}$, to which units may be attached. This is sometimes referred to as the value of the ratio. The ratio of 40 miles for every 5 hours has a unit rate of 8 miles per hour.

Word or Phrase	Definition
y-intercept	The *y*-intercept of a line is the *y*-coordinate of the point at which the line crosses the *y*-axis. It is the value of *y* that corresponds to $x = 0$. The *y*-intercept of the line $y = 3x + 6$ is 6. If $x = 0$, then $y = 6$. 

The Coordinate Plane

A coordinate plane is determined by a horizontal number line (the *x*-axis) and a vertical number line (the *y*-axis) intersecting at the zero on each line. The point of intersection (0, 0) of the two lines is called the origin. Points are located using ordered pairs (*x*, *y*).

- The first number (*x*-coordinate) indicates how far the point is to the right or left of the *y*-axis.
- The second number (*y*-coordinate) indicates how far the point is above or below the *x*-axis.



Point, coordinates, and interpretation

O (0, 0) → This is the intersection of the axes (origin).

P (2, 1) → start at the origin, move 2 units right, then 1 unit up

R (-3, -1) → start at the origin, move 3 units left, then 1 unit down

S (1, -3) → start at the origin, 1 unit right, then 3 units down

Q (-2, 0) → start at the origin, move 2 units left, then 0 units up or down

T (0, -2) → start at the origin, 0 units right or left, then 2 units down

Functions

Some ways to represent rules in mathematics are input-output tables, mapping diagrams, ordered pairs, equations, and graphs.

Examples that are Functions

Input-Output Table

x **input**	***y*** **output**
1	1
3	3
5	5
7	7
9	9

This table lists input values with unique output values.

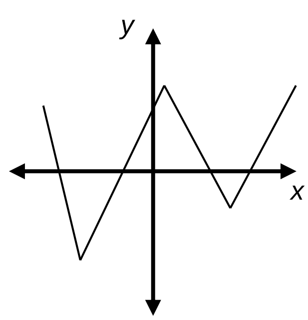
Ordered Pairs

(0, 2), (1, -2), (2, 2), (3, -2)

In this set of ordered pairs, each input value is assigned to a unique output value. Note that different input values may be assigned the same output value. In this example, both 1 and 3 are assigned the output value -2.

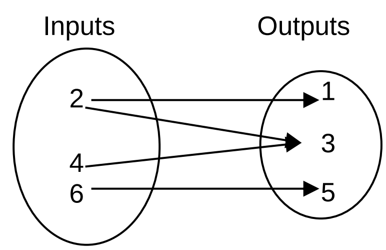
Graph

This graph represents a function because every vertical line through it intersects at most one point of the graph. In other words, each possible *x*-value corresponds to a unique *y*-value.



Examples that are NOT Functions

Mapping Diagram



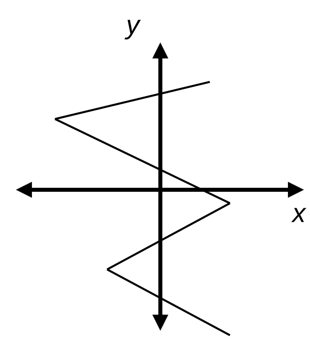
This mapping diagram is not a function. It is not permissible for the same input value (in this case 2) to be assigned two different output values. However, all other input-output mappings above are fine.

Equation (with Ordered Pairs)

Consider the set of pairs (x, y) that satisfy $x = y^2$, such as (0, 0), (25, 5), and (25, -5). Since the input value, x = 25, corresponds to two different output values (y = 5 and y = -5), the y-values are not a function of the x values.

Graph

This graph does not represent a function because some vertical lines (for example, the *y*-axis) intersect the graph in more than one point. In other words, some *x*-values correspond to more than one *y*-value.



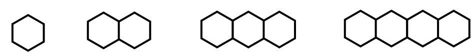
Using Multiple Representations to Describe Linear Functions

Here are four representations commonly used to approach a math problem:

- Numbers (numerical approach, as by making a table)
- Pictures (visual approach, as with a picture or graph)
- Symbols (approaching the problem using algebraic symbols)
- Words (verbalizing a solution, orally or in writing)

Each approach may lead to a valid solution. Collectively they should lead to a complete and comprehensive solution, one that is readily accessible to more people and that provides more insight.

Example 1: Describe this pattern of hexagons using numbers, pictures, words, and symbols.



Numbers

Step #	number of segments	Breaking apart numbers sometimes helps you see an input-output rule.	
1	6	6	= 6 + (0)5
2	11	6 + 5	= 6 + (1)5
3	16	6 + 5 + 5	= 6 + (2)5
4	21	6 + 5 + 5 + 5	= 6 + (3)5
5	26	6 + 5 + 5 + 5 + 5	= 6 + (4)5
n	5*n* + 1	5*n* + 1	= 6 + (*n* – 1)5

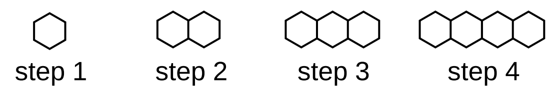
Words

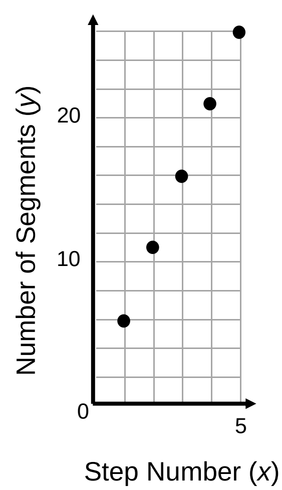
One way to describe the hexagonal pattern is to start with 6 segments and add 5 more segments at each subsequent step. Notice that the number of 5's added at each step is equal to 1 less than the step number.

Symbols

A rule for finding the number of segments at step *n* is 6 + (*n* – 1)5, which can be simplified to 5*n* + 1.

Pictures





Note: we consider a graph to be a picture.

Using Multiple Representations to Describe Linear Functions (Continued)

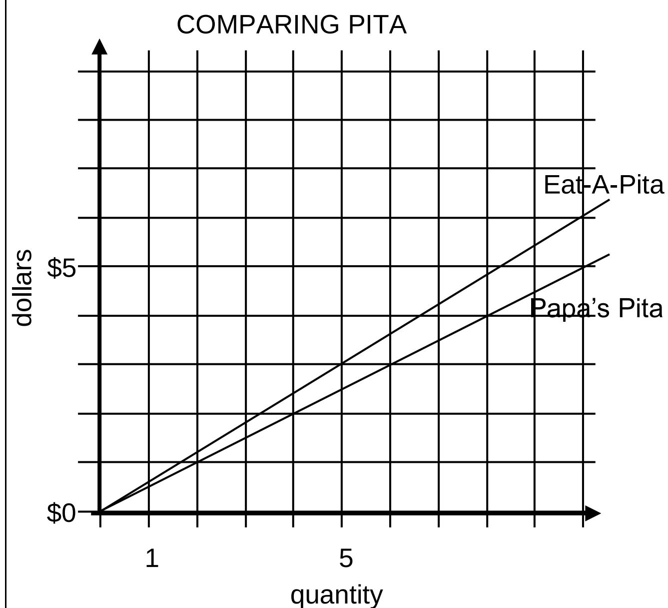
Example 2: At Papa's Pitas, 2 pitas cost $1.00. At Eat-A-Pita, 5 pitas cost $3.00. Assuming a proportional relationship between the number of pitas and their cost, use multiple representations to explore which store offers the better buy for pitas.

Numbers (make a table)

PAPA'S PITAS	
# of pitas (x)	cost (y)
2	$1.00
4	$2.00
6	$3.00
8	$4.00
10	$5.00

EAT-A-PITA	
# of pitas (x)	cost (y)
5	$3.00
10	$6.00
15	$9.00
20	$12.00
25	$15.00

Pictures (make a graph)



Words (write sentences)

Based on the table, Papa's Pitas is the better buy.

At Papa's Pitas, you get 6 pitas for $3.00.
This means the unit price (cost for one pita) is $0.50.

At Eat-A-Pita you only get 5 pitas for $3.00. This means the unit price (cost for one pita) is $0.60.

Symbols (write equations to relate the number of pitas to cost)

PAPA'S PITAS $y = 0.5x$

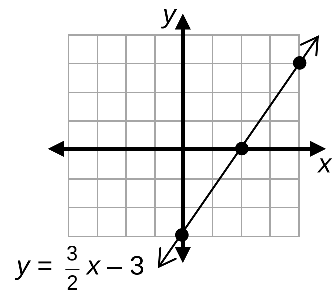
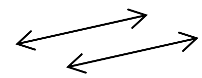
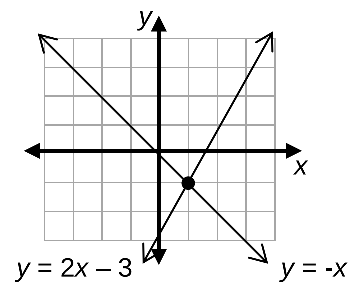
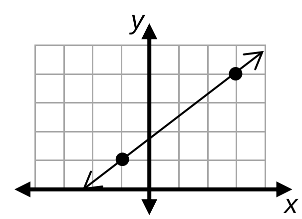
EAT-A-PITA $y = 0.6x$

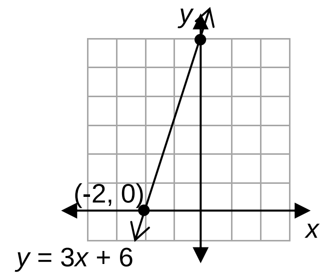
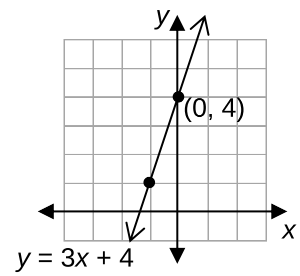
Notice that $0.50 is the cost of one pita at Papa's Pita. This corresponds to the point (1, 0.5) on the graph.

Notice that $0.60 is the cost of one pita at Eat-A-Pita. This corresponds to the point (1, 0.6) on the graph.

The equations above are both in the form $y = mx$. This equation form represents a proportional relationship because y is a constant multiple of x. Graphs of equations in this form are always lines going through the origin. They will be explored more in the next unit and contrasted with equations in the form $y = mx + b$.

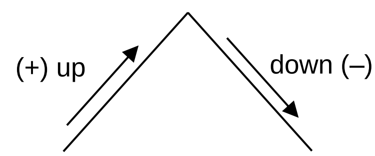
STUDENT RESOURCES

Word or Phrase	Definition
linear function	A <u>linear function</u> (in variables x and y) is a function that can be expressed in the form $y = mx + b$. The graph of $y = mx + b$ is a straight line with slope m and y-intercept b. The graph of the linear function $y = \frac{3}{2}x - 3$ is a straight line with slope $m = \frac{3}{2}$ and y-intercept $b = -3$. 
parallel	Two lines in a plane are <u>parallel</u> if they do not meet. 
point of intersection	A <u>point of intersection</u> of two lines is a point where the lines meet. The two straight lines in the plane with equations $y = -x$ and $y = 2x - 3$ have point of intersection (1, -1). 
slope-intercept form	The <u>slope-intercept form</u> of the equation of a line is the equation $y = mx + b$, where m is the slope of the line, and b is the y-intercept of the line. The equation $y = 2x + 3$ determines a line with slope 2 and y-intercept 3.
slope of a line	The <u>slope of a line</u> is the vertical change (change in the y-value) per unit of horizontal change (change in the x-value). If the difference in x is 0, we consider the slope to be undefined, a graphical representation of this situation is a vertical line. The slope of the line through (-1, 1) and (3, 4) is $\frac{3}{4}$: $\text{slope} = \frac{(\text{difference in } y)}{(\text{difference in } x)} = \frac{4-1}{3-(-1)} = \frac{3}{4}$ 

Word or Phrase	Definition
x-intercept	The <u>*x*-intercept</u> of a line is the *x*-coordinate of the point at which the line crosses the *x*-axis. It is the value of x that corresponds to $y = 0$. The *x*-intercept of the line $y = 3x + 6$ is -2. If $y = 0$, then $x = -2$. 
y-intercept	The <u>*y*-intercept</u> of a line is the *y*-coordinate of the point at which the line crosses the *y*-axis. It is the value of y that corresponds to $x = 0$. For the line $y = 3x + 4$, the *y*-intercept is 4. If $x = 0$, then $y = 4$. 

Slope

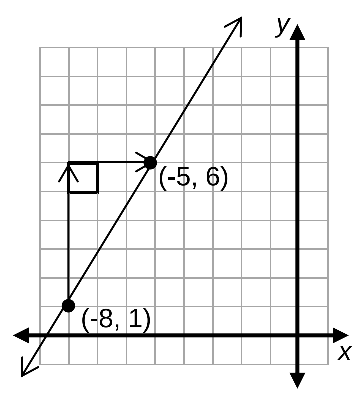
One way to think about slope (m) is to imagine that the line is a portion of a mountain. Just as we read from left to right, we will move up and down the mountain from left to right. When moving up the mountain, the slope is positive. When moving down the mountain, the slope is negative. The steeper the mountain, the greater (in absolute value) the slope.



The slope (m) of a line is computed as: $\frac{\text{vertical change}}{\text{horizontal change}}$ as you move from one point to another on the same line, or $\frac{\text{difference in } y \text{ coordinates}}{\text{difference in } x \text{ coordinates}}$ as you move from one point to another on the same line.

To use counting to determine slope, first move in a vertical direction and find the directed distance, and then move in a horizontal direction and find the directed distance.

If A (-8, 1) and B (-5, 6) are points on a line, then count 5 units up and then 3 units to the right. $m = \frac{5}{3}$



To use coordinates to determine slope (m), find the quotient of the difference in the *y*-coordinates and the difference in the *x*-coordinates.

If A (-8, 1) and B (-5, 6) are points on a line, then

$$m = \frac{\text{difference in } y}{\text{difference in } x} = \frac{6-1}{-5-(-8)} = \frac{5}{3}.$$

If (a, b) and (c, d) are points on a line, then

$$m = \frac{\text{difference in } y}{\text{difference in } x} = \frac{d-b}{c-a}.$$ This formula is the definition of the slope of a line.

Horizontal and Vertical Lines

The slope (m) of a line is computed as:

$$\frac{\text{difference in } y \text{ coordinates}}{\text{difference in } x \text{ coordinates}}$$ as you move from one point to another on the same line.

Horizontal Lines

A horizontal line is a line parallel to the x-axis. Every point on a horizontal line has the same y-coordinate, and the vertical change between any two positions on the line is zero. Hence,

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{0}{\text{horizontal change}} = 0.$$

horizontal

The slope of a horizontal line is zero.

Vertical Lines

vertical

A vertical line is a line parallel to the y-axis. Every point on a vertical line has the same x-coordinate, and the horizontal change between any two points on the line is zero. Hence,

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{vertical change}}{0}$$ is undefined,

since division by zero is undefined.

The slope of a vertical line is undefined.

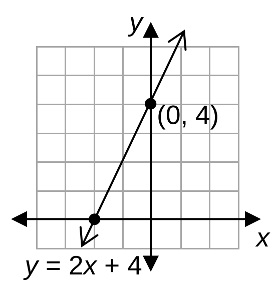
The Slope-Intercept Form of Linear Equations

<u>Slope-intercept form of a linear equation</u> is $y = mx + b$, where m = slope of the line and b = the y-intercept.

Find the equation of a line with a slope of $-\frac{1}{3}$ and the y-intercept is -5.

Since $y = mx + b$, then $y = -\frac{1}{3}x - 5$.

Find the equation of the line that passes through the points (0, 4) and (-2, 0).

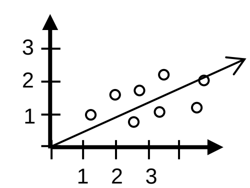


First plot the points on a graph.
Notice that the y-intercept is 4.
Count or compute to find the slope,

$$m = \frac{4 - 0}{0 - (-2)} = 2$$

Therefore, the equation of the line is $y = 2x + 4$.

STUDENT RESOURCES

Word or Phrase	Definition
association	In statistics, an association between two variables is a relationship between the variables, so that the variables are statistically dependent. In the case of numerical variables, if the relationship is linear, we refer to a linear association between the variables.
bivariate data	Bivariate data is data that has two variables. Bivariate data can be represented by ordered pairs. A list of country of origin and batting average for each baseball player is a bivariate data set with one categorical variable and one numerical variable.
bivariate numerical data	Bivariate numerical data is data that has two numerical variables. Bivariate numerical data can be represented by a scatter plot, so that the relationship (if any) between the variables is more easily seen. A list of heights and weights for each player on a football team is a bivariate numerical data set.
categorical data	Categorical data is data sorted into categories, such as colors, ranges of measurements, or other attributes of the data. Generally, there are only finitely many categories.
data set	A data set is a collection of pieces of information about a population, often numbers, obtained from observation, questioning, or measuring.
frequency table	A frequency table is a table that lists items and the number of times they occur in a data set.
line of best fit	A line of best fit for a scatter plot is a straight line that best represents (in some sense) the data points in the scatter plot. 
measurement data	Measurement data is numerical data that comes from making measurements. Measurement data can be obtained by measuring such things as heights, weights, temperatures, lengths, areas, and volumes.
numerical data	Numerical data is data consisting of numbers. The numbers allow for statistical calculations, such as finding the mean or median.
outlier	An outlier of a data set is a data value that is unusually small or unusually large relative to the overall pattern of values in the data set. For the data set {1, 1, 1, 3, 5, 6, 6, 7, 23}, the data value 23 is a potential outlier.

Word or Phrase	Definition
population	In statistics, the population refers to the source of a data set. If we wish to make statistical inferences about the students at a school, we may take a random sample of the students, or we may gather data from all the students. In either case, the population refers to the students in the school.
relative frequency table	A relative frequency table is a frequency table that lists items and the proportion (or percent) of times they occur.
statistical question	A statistical question is a question where numerical data that has potential for variability can be collected and analyzed for the purpose of answering the question. A statistical question: "How much TV do middle school students watch on average?" NOT a statistical question: "How many hours of TV did you watch last week?"
two-way table	A two-way table is a table that displays bivariate categorical data, in which the rows correspond to the categories of one variable, and the columns correspond to the categories of the other. A two-way table that includes the number of data observations is called a "two-way frequency table". A two-way table that includes the percentage of the number of data observations relative to the total number of observations is called a "two-way relative frequency table".

Numerical Data

Numerical data is data consisting of numbers. Measurement data is numerical data that comes from making measurements.

Numerical survey questions are used to collect numerical data. Numerical data typically come from counting or measurements. Examples of numerical survey questions include:
- How many dogs do you own? (a counting question)
- How many minutes did you exercise last week? (a measurement question)

Some ways to report one-variable (or univariate) numerical data include:
- Measures of center such as mean, median, mode
- Measures of spread such as range, mean absolute deviation (MAD), and 5-number summary
- Data displays such as tables, line plots, histograms, and box plots

Some ways to report two-variable (or bivariate data) numerical data include:
- Tables
- Graphs
- Equations

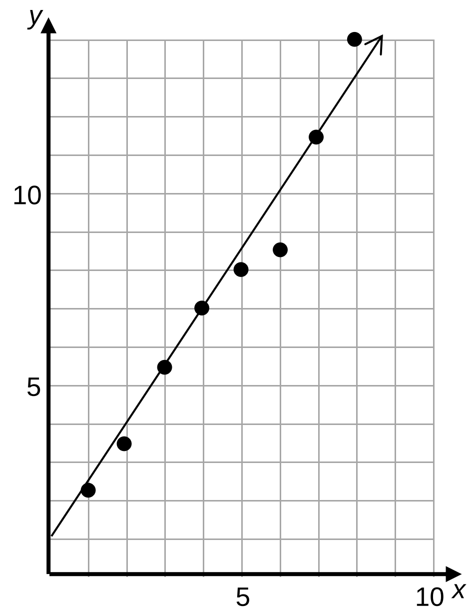
Lines of Best Fit

A line of best fit for a scatter plot is a straight line that best represents (in some sense) the data points in the scatter plot.

Example: When the data in the table below is graphed in a scatter plot, the data points cluster along a straight line. We conclude that there is likely a linear association between x and y. One possible such line may be estimated by the equation graphed below, $y = \frac{3}{2}x + 1$.

Using a graphing calculator, another estimated equation is given as $y = 1.6x + 0.3$ (not graphed).

x	1	2	3	4	5	6	7	8
y	2.2	3.5	5.5	7	8	8.5	11.5	14

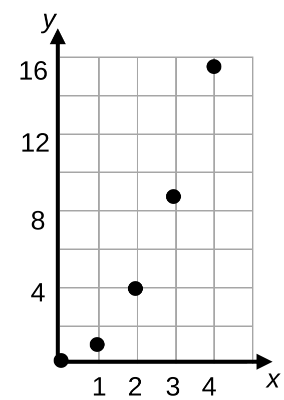


Nonlinear Associations

Not all associations are linear. Here is an example of a scatter plot of bivariate data that appears to have a nonlinear association.

Example: For this data set, the graphed points do not fall in a linear pattern. They increase at an increasing rate.

x	0	1	2	3	4
y	0	1	4	9	16



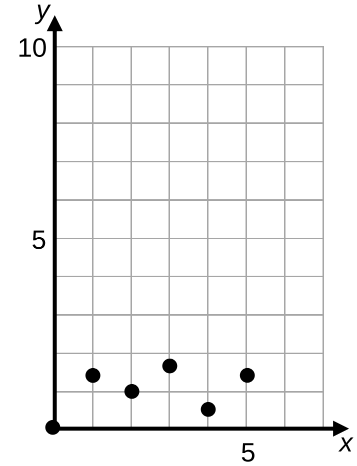
Outliers

An <u>outlier</u> of a data set is a data value that is unusually small or unusually large relative to the overall pattern of values in the data set.

Outliers can create the illusion that an association exists when one does not. They can also distract us from seeing an association when there clearly is one.

Example 1: In the scatter plot to the right, the data point (6, 10) is a potential outlier. Its y-coordinate 10 appears to be unusually large compared to the other y-coordinates.

Example 2: In a 6th grade classroom, students were asked how many pets they had. All students but one replied with numbers of pets that ranged from 0 to 8. That one pet owner said she had 40 fish. This number of fish appears to be an outlier, because it is unusually large compared to the other numbers of pets.



Categorical Data

<u>Categorical data</u> is data sorted into categories, such as colors, ranges of measurements, or other attributes of the data. Generally, there are only finitely many categories.

Categorical survey questions are used to collect categorical data. Responses to these questions are usually in words. Examples of categorical survey questions include:

- What types of pets do you own? (Answers include dog, cat, bird, no pets, etc.)
- Do you have a curfew? (A yes-no answer)

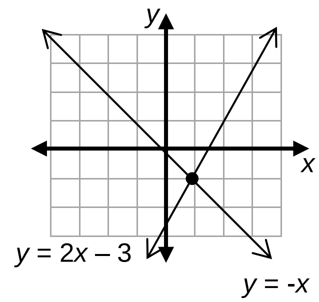
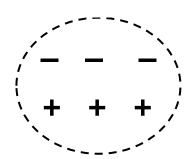
Some ways to report one-variable categorical data include

- Frequency tables
- Relative frequency tables
- Pie charts (circle graphs)
- Bar graphs

Some ways to report two-variable categorical data include:

- Two-way frequency tables
- Two-way relative frequency tables

STUDENT RESOURCES

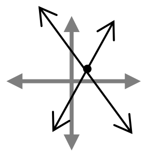
Word or Phrase	Definition
point of intersection	A <u>point of intersection</u> of two lines is a point where the lines meet. The two straight lines in the plane with equations $y = -x$ and $y = 2x - 3$ have point of intersection (1, -1). 
slope-intercept form	The <u>slope-intercept form</u> of the equation of a line is the equation $y = mx + b$, where m is the slope of the line, and b is the y-intercept of the line. The equation $y = 2x + 3$ determines a line with slope 2 and y-intercept 3.
solution to an equation	A <u>solution to an equation</u> involving variables consists of values for the variables which, when substituted, make the equation true. The value $x = 8$ is a solution to the equation $10 + x = 18$. If we substitute 8 for x in the equation, the equation becomes true: $10 + 8 = 18$.
solve an equation	To <u>solve an equation</u> refers to finding all values for the variables in the equation that, when substituted, make the equation true. Values that make an equation true are called solutions to the equation. To solve the equation $2x = 6$, one might think "two times what number is equal to 6?" Since 2(3) = 6, the only value for x that satisfies this condition is 3. Therefore 3 is the solution.
substitution	<u>Substitution</u> refers to replacing a value or quantity with an equivalent value or quantity. If $y = x + 5$, and we know that $x = 3$, then we may use substitution to rewrite the first equation to get $y = 3 + 5$. If $y = x + 10$, and we know also that $y = 2x + 4$, then we may use substitution to write one equation in x to get $x + 10 = 2x + 4$.
system of linear equations	A <u>system of linear equations</u> is a set of two or more linear equations in the same variables. An example of a system of linear equations in x and y: $\begin{cases} x + y = 1 \\ x + 2y = 4 \end{cases}$
zero pair	In the signed counters model, a positive and a negative counter together form a <u>zero pair</u>. Let "+" represent a positive counter, and let "–" represent a negative counter. Then the following is an example of a collection of (three) zero pairs. 

Systems of Linear Equations

A system of equations is a set of two or more equations in the same variables. A solution to the system of equations consists of values for the variables which, when substituted, make all equations simultaneously true.

A system of linear equations has exactly one solution, infinitely many solutions, or no solution.

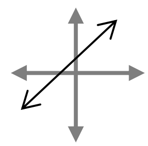
$$\begin{cases} y = -x + 3 \\ y = 2x - 3 \end{cases}$$



One solution at (2, 1)

These two lines intersect in aqexactly one point.This is the only pair of *x*- and *y*-values that statisfies these two equations simultaneously.

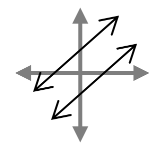
$$\begin{cases} y = x + 1 \\ 2y = 2x + 2 \end{cases}$$



Infinitely many solutions

Since the equations are equivalent, the two lines coincide. Every point on the line represents a solution.

$$\begin{cases} y = x - 2 \\ y = x + 2 \end{cases}$$



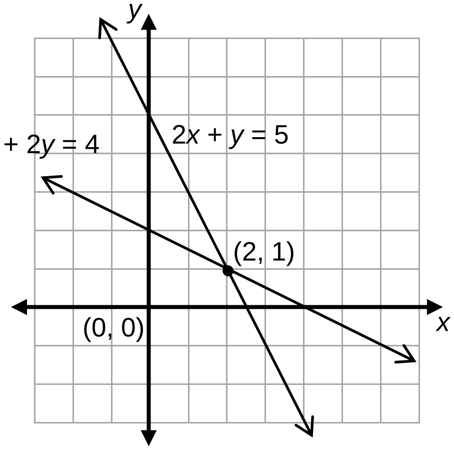
No solution

Since these two lines are parallel, they do not intersect. Thus these two equations have no solution in common.

Solving a System of Linear Equations by Graphing

To solve a system of equations by graphing, graph both lines on the same set of axes and observe the point(s) of intersection, if any.

Solve by graphing: $\begin{cases} 2x + y = 5 \\ x + 2y = 4 \end{cases}$



1. Change each to slope-intercept form, $y = mx + b$.

   $2x + y = 5 \longrightarrow y = -2x + 5$

   $x + 2y = 4 \longrightarrow y = -\frac{1}{2}x + 2$

2. Graph each equation.

3. Observe the intersection of the lines, (2, 1). This represents the solution to the system. In other words, these are the x- and y- values that satisfy both equations. Remember that not every system of equations has exactly one solution.

4. Check by substituting solutions in the original equations to be sure they are correct.

   $2x + y = 5 \longrightarrow 2(2) + 1 = 5$ (true)

   $x + 2y = 4 \longrightarrow 2 + 2(1) = 4$ (true)

Properties of Arithmetic

Properties of arithmetic govern the manipulation of expressions (mathematical phrases).
For any three numbers a, b, and c:

- ✓ Associative property of addition
  $a + (b + c) = (a + b) + c$
- ✓ Commutative property of addition
  $a + b = b + a$
- ✓ Additive identity property
  (addition property of 0)
  $a + 0 = 0 + a = a$
- ✓ Additive inverse property
  $a + (-a) = -a + a = 0$
- ✓ Associative property of multiplication
  $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
- ✓ Commutative property of multiplication
  $a \bullet b = b \bullet a$
- ✓ Multiplicative identity property
  (multiplication property of 1)
  $a \bullet 1 = 1 \bullet a = a$
- ✓ Multiplicative inverse property
  $a \bullet \frac{1}{a} = \frac{1}{a} \bullet a = 1 \quad (a \neq 0)$
- ✓ Distributive property relating addition and multiplication
  $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c.

Properties of Equality

Properties of equality govern the manipulation of equations (mathematical sentences).
For any three numbers a, b, and c:

- ✓ Addition property of equality
  (Subtraction property of equality)
  If $a = b$ and $c = d$, then $a + c = b + d$
- ✓ Multiplication property of equality
  (Division property of equality)
  If $a = b$ and $c = d$, then $ac = bd$
- ✓ Reflexive property of equality: $a = a$
- ✓ Symmetric property of equality: If $a = b$, then $b = a$
- ✓ Transitive property of equality: If $a = b$, and $b = c$, then $a = c$

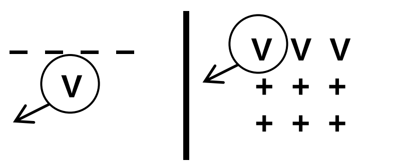
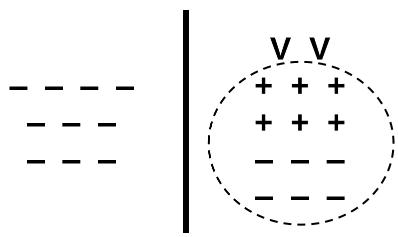
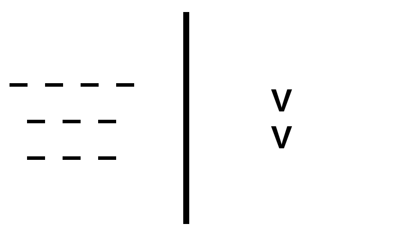
Solving Equations Using a Model 1

Let **+** represent 1 | Let **V** represent the unknown (like *x*)

Let **–** represent -1 | Let **Λ** represent the opposite of the unknown (like -*x*)

The following example illustrates one solution path. Other paths are possible to arrive at the same solution.

Solve: $-4 + x = 3(x + 2)$

Picture	Equation	What did you do?
	$-4 + x = 3x + 6$ $\underline{-x} \quad \underline{-x}$ $-4 = 2x + 6$	build the equation remove one *x* from each side
	$-4 = 2x + 6$ $+(-6) \quad +(-6)$ $-10 = 2x$	add -6 to each side remove zero pairs
	$\frac{-10}{2} = \frac{2}{2}x$ $-5 = x$	divide both sides by 2 put counters equally into cups (or do mentally) notice the use of the big 1

Check by substituting the solution into the original equation:

$-4 + x$	=	$3(x + 2)$	
$-4 + (-5)$	=	$3(-5 + 2)$	?
-9	=	$3(-3)$	?
-9	=	-9	true

Solving Equations Using a Model 2

Let **+** represent 1

Let **–** represent -1

Let **V** represent the unknown (like *x*)

Let **Λ** represent the opposite of the unknown (like -*x*)

The following example illustrates one solution path. Other paths are possible to arrive at the same solution.

Solve: $-2x - 1 = x - 4$

Picture	Equation	What did you do?
Λ Λ – | V – – – –	$-2x - 1 = x - 4$	build the equation
Λ Λ Λ – | V Λ – – – –	$-2x - 1 = x - 4$ $+(-x) = +(-x)$ $-3x - 1 = -4$	add the opposite of *x* to both sides remove the zero pair
Λ Λ Λ – | – – – –	$-3x - 1 = -4$ $-(-1) = -(-1)$ $-3x = -3$	remove -1 from both sides* *this gives the same result as adding 1 to each side
Λ Λ Λ V V V + + + | – – – + + + V V V	$-3x = -3$ $(+3x) + 3 = +3 + (3x)$ $3 = 3x$	add 3*x* to both sides AND add 3 positives to both sides remove zero pairs
+ + + | V V V	$\frac{3}{3} = \frac{3}{3}x$ $1 = x$	divide both sides by 3 put counters equally into cups (or do mentally) notice the use of the big 1

Check by substituting the solution into the original equation:

$-2x - 1$	=	$x - 4$	
$-2(1) - 1$	=	$1 - 4$	?
$-2 - 1$	=	-3	?
-3	=	-3	true

Using Algebraic Techniques to Solve Equations
To solve equations using algebra: • Use the properties of arithmetic to simplify each side of the equation (e.g., associative properties, commutative properties, inverse properties, distributive property). • Use the properties of equality to isolate the variable (e.g., addition property of equality, multiplication property of equality).
Solve: $3 - x + 3 = 5x - 2x - 2$

Equation	What did you do?	Property
$3 - x + 3 = 5x - 2x - 2$ $6 - x = 3x - 2$	arithmetic collect like terms	distributive property $(5 - 2)x = 3x$
$6 - x = 3x - 2$ $+ 2 \qquad + 2$ $8 - x = 3x$	add 2 to both sides arithmetic	addition property of equality additive inverse/identity properties
$8 - x = 3x$ $+ x \quad + x$ $8 = 4x$	add x to both sides collect like terms	addition property of equality additive inverse/identity properties distributive property $3x + x = (3 + 1)x = 4x$
$\frac{8}{4} = \frac{4x}{4}$ $2 = x$	multiply both sides by $\frac{1}{4}$ (or divide both sides by 4) arithmetic	multiplication(division) property of equality multiplicative inverse/identity properties

Check by substituting the solution into the original equation:

$3 - x + 3$	=	$5x - 2x - 2$	
$3 - 2 + 3$	=	$5(2) - 2(2) - 2$	?
4	=	$10 - 4 - 2$	?
4	=	4	true

STUDENT RESOURCES

Word or Phrase	Definition
slope-intercept form	The slope-intercept form of the equation of a line is the equation $y = mx + b$, where m is the slope of the line, and b is the y-intercept of the line. The equation $y = 2x + 3$ determines a line with slope 2 and y-intercept 3.
solution to an equation	A solution to an equation involving variables consists of values for the variables which, when substituted, make the equation true. The value $x = 8$ is a solution to the equation $10 + x = 18$. If we substitute 8 for x in the equation, the equation becomes true: $10 + 8 = 18$.
solve an equation	To solve an equation refers to finding all values for the variables in the equation that, when substituted, make the equation true. Values that make an equation true are called solutions to the equation. To solve the equation $2x = 6$, one might think "two times what number is equal to 6?" Since $2(3) = 6$, the only value for x that satisfies this condition is 3. Therefore 3 is the solution.
substitution	Substitution refers to replacing a value or quantity with an equivalent value or quantity. If $y = x + 5$, and we know that $x = 3$, then we may use substitution to rewrite the first equation to get $y = 3 + 5$. If $y = x + 10$, and we know also that $y = 2x + 4$, then we may use substitution to write one equation in x to get $x + 10 = 2x + 4$.
system of linear equations	A system of linear equations is a set of two or more linear equations in the same variables. An example of a system of linear equations in x and y: $\begin{cases} x + y = 1 \\ x + 2y = 4 \end{cases}$

Properties of Arithmetic

Properties of arithmetic govern the manipulation of expressions (mathematical phrases).
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- ✓ Associative property of addition
  $a + (b + c) = (a + b) + c$
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  $a + (-a) = -a + a = 0$
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  $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
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- ✓ Multiplicative identity property
  (multiplication property of 1)
  $a \bullet 1 = 1 \bullet a = a$
- ✓ Multiplicative inverse property
  $a \bullet \frac{1}{a} = \frac{1}{a} \bullet a = 1 \quad (a \neq 0)$
- ✓ Distributive property relating addition and multiplication
  $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers *a*, *b*, and *c*.

Properties of Equality

Properties of equality govern the manipulation of equations (mathematical sentences).
For any three numbers *a*, *b*, and *c*:

- ✓ Addition property of equality
  (Subtraction property of equality)
  If $a = b$ and $c = d$, then $a + c = b + d$
- ✓ Multiplication property of equality
  (Division property of equality)
  If $a = b$ and $c = d$, then $ac = bd$
- ✓ Reflexive property of equality: $a = a$
- ✓ Symmetric property of equality: If $a = b$, then $b = a$
- ✓ Transitive property of equality: If $a = b$, and $b = c$, then $a = c$

A Strategy for Solving Equations with Rational Coefficients	
Equations with rational number coefficients may be solved the same way that equations with integer coefficients are solved using properties of arithmetic and equality. However, many people prefer to rewrite the equation without fractions or decimals before solving it. This can be accomplished by: • Determining a common multiple for all denominators in the equation. • Multiplying both sides of the equation by that common multiple. What remains will be an equation to solve with integer coefficients.	
$\frac{2}{3}(x-1) = \frac{1}{6}(x+2)$ $6\left[\frac{2}{3}(x-1)\right] = 6\left[\frac{1}{6}(x+2)\right]$ $4(x-1) = x+2$	A common multiple of 3 and 6 is 6. Here we multiply both sides of the equation by 6 (multiplication property of equality). The result is an equation with integer coefficients.
$0.1x + 0.25(33-x) = 5.1$ $100\ [0.1x + 0.25(33-x)] = 100\ [5.1]$ $10x + 25(33-x) = 510$	This equation includes tenths ($\frac{1}{10}$) and hundredths ($\frac{1}{100}$). A common multiple of these denominators is 100. Here we multiply both sides of the equation by 100. The result is an equation with integer coefficients.

Solving a System of Linear Equations by Substitution

While there may be multiple ways to use substitution to solve a system of equations, this is one way that has been demonstrated in this unit.

1. Write both equations in slope-intercept form.

$$\begin{cases} 2x + y = 5 \quad \rightarrow \quad y = -2x + 5 \\ x + 2y = 4 \quad \rightarrow \quad 2y = -x + 4 \quad \rightarrow \quad y = -\frac{1}{2}x + 2 \end{cases}$$

2. Use the substitution property. Since the right side of both equations are equal to the same thing, namely *y*, the two expressions in *x* must be equal to each other.

   Write one equation and solve for *x*.

$$-2x + 5 = -\frac{1}{2}x + 2$$
$$x = 2$$

3. Substitution this *x*-value into either equation to obtain the *y*-value.

$$y = -2(2) + 5$$
$$y = 1$$

Solution to the system: (2, 1)

For this example, another substitution approach would be to write the first one equation in slope-intercept form and substitute the expression for *y* in the second equation. Using this approach:

$$x + 2(-2x + 5) = 4$$
$$x = 2$$

Solving a System of Linear Equations by Elimination

Elimination, which applies the multiplication property of equality and the addition property of equality, is another method for solving a system of equations. Here is an example.

Example: Solve this system of equations by elimination.
(We will number each equation with brackets to keep track of them.)

$$\begin{cases} 2x + y = 5 & [1] \\ x + 2y = 4 & [2] \end{cases}$$

1. Use the multiplication property of equality. Multiply both sides of one (or both) equations by some number that will make one of the variable expressions in each equation opposites of each other. In this case, we might multiply both sides of the first equation by -2.

$$-2(2x + y) = -2(5) \rightarrow -4x - 2y = -10 \quad [3]$$

2. Use the addition property of equality. Add expressions on each side of the equation together. Solve.

$$\begin{array}{rcll} -4x - 2y & = & -10 & [3] \\ \underline{x + 2y} & = & \underline{4 \quad\quad [2]} & \\ -3x & = & -6 & \\ x & = & 2 & \end{array}$$

3. Substitute into one of the original equations to find y.

$$[1] \quad 2x + y = 5 \rightarrow 2(2) + y = 5$$

$$y = 1$$

4. Substitute into the other original equation to check.

$$[2] \quad x + 2y = 4 \quad 2 + 2(1) = 4 \quad \text{(true)}$$

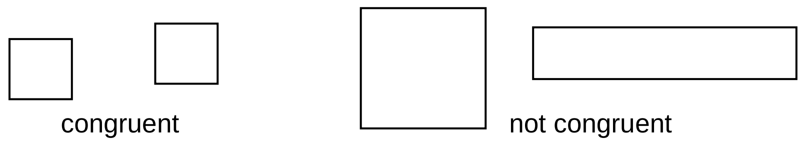
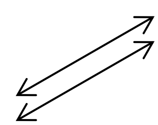
A Strategy for Organizing Problem Solving Work Involving Equations

Many algebra problems can be solved with these steps. Here are the steps and an example.

> Talia's coin jar contains nickels and quarters. There are 38 coins in all. The total value of jar is $5.30. Find the coins in the jar.

- Identify the variable(s) — Let n = the number of nickels
  Let q = the number of quarters
- Write the equation(s) — $n + q = 38 \rightarrow q = 38 - n$
  $0.05n + 0.25q = 5.30$
- Solve the equation(s) — By substitution:
  $0.05n + 0.25(38 - n) = 5.30$
  $n = 21$ and $q = 17$
- Answer the question(s) — There are 21 nickels and 17 quarters
- Interpret the solution in the problem — Money makes sense: 21($0.05) + 17($0.25) = $5.30

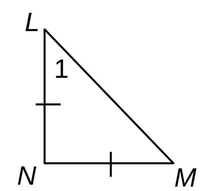
STUDENT RESOURCES

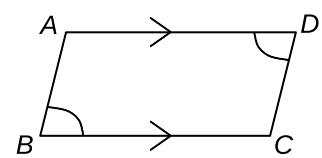
Word or Phrase	Definition
congruent figures	Two figures in the plane are congruent figures if the second can be obtained from the first by a sequence of one or more of translations, rotations, and reflections. Two squares are congruent if they have the same side-length.  congruent not congruent If $\triangle ABC$ is congruent to $\triangle DEF$, we write $\triangle ABC \cong \triangle DEF$.
function	A function is a rule that assigns to each input value exactly one output value. A function may also be referred to as a transformation or mapping. The collection of output values is the image of the function. Consider the function $y = 3x + 6$. For any input value, say $x = 10$, there is a unique output value, in this case $y = 36$. This output value is obtained by substituting the value of x into the equation. This function represents a straight line consisting of all ordered pairs of points (x, y) that satisfy the equation. Consider the transformation $(x, y) \rightarrow (-x, y)$. This transformation maps the x-coordinates in the plane to their opposites, while y-coordinates remain the same. In this case the image is a reflection about the y-axis.
image	The image of a function or transformation is the collection of its output values. The input values are then referred to as the pre-image. See transformation.
parallel	Two lines in a plane are parallel if they do not meet. Two line segments in a plane are parallel if the lines they lie on are parallel. 
reflection	A reflection of a plane through a line L is the transformation that maps each point to its mirror image on the other side of L. The line L is called the line of reflection. The transformation $(x, y) \rightarrow (x, -y)$ is a reflection of the plane through the x-axis.
rigid motion	A rigid motion is a transformation that preserves distances. Any rigid motion of the plane is a sequence of one or more translations, rotations, and reflections. Rigid motions also preserve lengths, angle measures, and parallel lines.
rotation	A rotation of a plane is a transformation that turns it through a given angle about a given point. The given angle is called the angle of rotation, and the given point is called the center point of rotation. The transformation $(x,y) \rightarrow (-y,x)$ is a rotation of the plane about the origin through angle 90°.

Word or Phrase	Definition
transformation	A <u>transformation</u> is a function that maps points in the plane (called the pre-image) to points in the plane (called the image). Rigid motion transformations include translations, rotations, and reflections.
translation	A <u>translation</u> of the plane is the transformation of the plane that maps pre-image points to image point in the same distance and direction. The transformation $(x, y) \rightarrow (x + 1, y + 2)$ slides all points 1 unit to the right and 2 units up.

Geometry Notation

These are examples of geometry diagrams and notations used in this program.





- A point is named using capital letters.

  Example: point *M*

- A polygon (e.g., triangle, parallelogram) is identified with a small symbol followed by its vertices.

  Examples: $\triangle LMN$, $\square ABCD$

- Line segment from point *L* to point *N is* named with the endpoints and a "bar" over them.

  Example: $\overline{LN}$

- The length (a measure) of a line segment from point *L* to point N is distinguished from the line segment (an object) by using absolute value symbols.

  Example: $\left|\overline{LN}\right|$

- An angle is named at its vertex and points on its rays, if needed.

  Example: The angle at *L* may be denoted $\angle L$, $\angle NLM$, $\angle MLN$, or $\angle 1$.

- The measure of an angle is distinguished from the angle (an object) using absolute value symbols.

  Example: The measure of $\angle L$ is written as $\left|\angle L\right|$.

- The symbol $||$ indicates parallel lines. Arrows in a diagram indicate parallel segments as well.

  Example: $\overline{AD} \,||\, \overline{BC}$

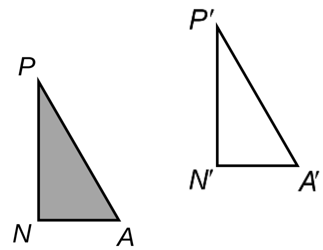
- The symbol $\cong$ indicates congruence. Tick marks on the diagram above indicate congruent segments and arcs indicate congruent angles as well.

  Examples: Line segments $\overline{LN}$ and $\overline{NM}$ have the same length. Therefore, $\overline{LN} \cong \overline{NM}$.

  Angles at *B* and *D* have the same measure. Therefore, $\angle B \cong \angle D$.

Transformations of the Plane

A <u>transformation</u> is a function that maps points in the plane (called the spre-image) to points in the plane (called the <u>image</u>).



The input values (called the pre-image) are points in the plane. The output values (called the <u>image</u> of the transformation) are also points in the plane.

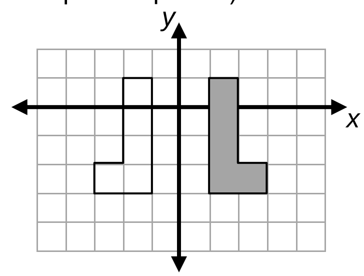
A transformation can be viewed as a <u>mapping</u> of pre-images (input values) to their corresponding images (output values).

In this figure, shaded triangle $\triangle PAN$ represents input values of a transformation and unshaded triangle $\triangle P'A'N'$ represents its image (output values).

The prime symbol (an apostrophe-like symbol) is often used to distinguish points in a pre-image (input values) from their images (output values).

We use the arrow notation $P \rightarrow P'$ (read “point *P* is taken to point *P* prime” or “*P* maps to *P* prime”) to indicate that the image of the point P under the transformation is P'.

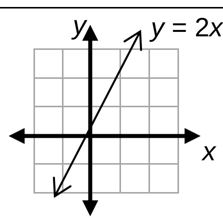
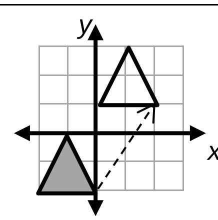
In a coordinate plane, we use the coordinates to describe the transformation as in the following example.



The reflection over the *y*-axis maps the shaded L-figure to a backwards L-figure. We use the arrow notation to describe this transformation. In this example, $(x, y) \rightarrow (-x, y)$.

Comparison of an Algebraic Function and a Geometric Function

Functions arise in many different contexts. The way we think of them and even the language we use to talk about them may be quite different for different areas of math. Here we compare a typical <u>function</u> we might meet in an algebra course and a typical function (we call it a <u>transformation</u>) that we might study in geometry.

Name of function	Linear function	Translation
Rule	Multiply by 2	Translate 2 units right and 3 units up
Description with symbols	x maps to $2x$ $x \rightarrow 2x$ $y = 2x$	(x, y) maps to $(x + 2, y + 3)$ $(x, y) \rightarrow (x + 2, y + 3)$
Graph		
Graph interpretation	The *x*-coordinates represent the inputs and the *y*-coordinates represent the outputs. The set of all input-output pairs is represented by the line.	A figure (shaded triangle - input) and its image (unshaded triangle - output) illustrate what happens to a typical figure in the plane. The translation arrow shows the direction and distance each point is moved.

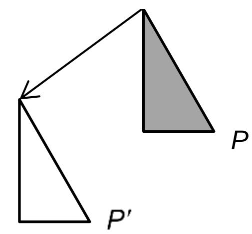
Translations, Rotations, and Reflections

Translations, rotations, and reflections are transformations of the plane that preserve distance between points.

A translation is a transformation that shifts all points the same distance and in the same direction.

This translation maps P to P' ($P \rightarrow P'$).
(read "P maps to P prime").

The translation arrow shows the shift

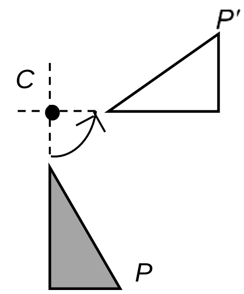


A rotation of a plane is a transformation that turns it through a given angle about a given point. The given point is called the center point of rotation. The given angle is called the angle of rotation.

This rotation maps P to P' ($P \rightarrow P'$).

Point C is the center point of the rotation.

The angle of rotation is 90° (or a quarter counterclockwise).

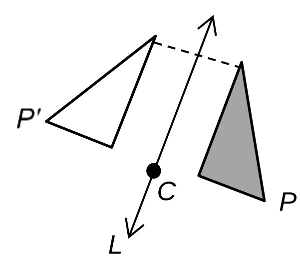


The reflection of a plane through a line L is the transformation that takes each point to its mirror image on the other side of L.

This reflection maps P to P' ($P \rightarrow P'$).

Line L is the line of reflection.

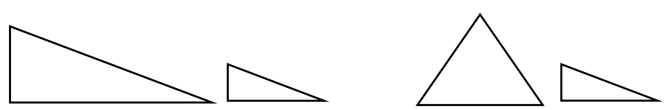
Line L is the perpendicular bisector of $\overline{PP'}$.



Translations, rotations, and reflections preserve distances between points. Further, translations, rotations, and reflections

- map lines to lines,
- map line segments to line segments of the same length,
- map parallel lines to parallel lines, and
- map angles to angles of the same measure.

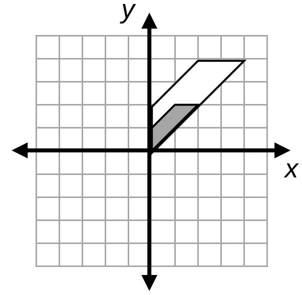
STUDENT RESOURCES

Word or Phrase	Definition
congruent figures	Two figures in the plane are congruent figures if the second can be obtained from the first by a sequence of one or more translations, rotations, and reflections. Two squares are congruent if they have the same side-length.  congruent not congruent
dilation	A dilation is a transformation that moves each point along the ray through the point emanating from a fixed center, multiplying distances from the center by a common scale factor. The fixed center is referred to as a “center point.” The transformation of the plane mapping $(x, y) \rightarrow (2x, 2y)$ is a dilation with center at the origin and scale factor 2.
image	The image of a function or transformation is the collection of its output values. The input values are then referred to as the pre-image. See transformation.
rigid motion	A rigid motion is a transformation that preserves distances. Any rigid motion of the plane is a sequence of one or more translations, rotations, and reflections. Rigid motions also preserve lengths, angle measures, and parallel lines.
scale factor	A scale factor is a positive number which multiplies some quantity.
similar figures	Two figures in the plane are similar figures if one can be moved to exactly cover the other by a sequence of one or more translations, rotations, reflections, and dilations. In similar figures, corresponding angles are congruent, and lengths of corresponding sides are proportional.  similar not similar If $\triangle ABC$ is similar to $\triangle DEF$, we write $\triangle ABC \sim \triangle DEF$.
transformation	A transformation is a function that maps points in the plane (called the pre-image) to points in the plane (called the image). Translations, rotations, reflections, and dilations are transformations of the plane.

Dilations of the Plane

A dilation is a transformation of the plane that is used to resize an object.
Multiplying by a scale factor:

- greater than 1 results in an enlargement of the pre-image.
- between 0 and 1 results in a reduction of the pre-image.
- equal to 1 results in a figure congruent to the pre-image.



The transformation of the plane to the right

- has a center at the origin and a scale factor of 2, and
- can be represented by the rule $(x, y) \rightarrow (2x, 2y)$.

Dilations share many but not all of the properties of translations, rotations, and reflections. Dilations

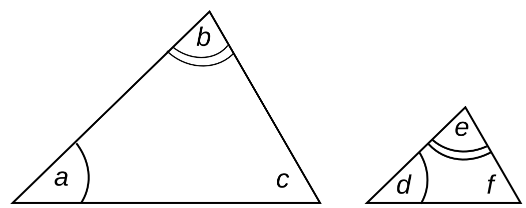
- map lines to lines (in fact to lines with the same slope),
- map parallel lines to parallel lines (that is, they preserve parallelism), and
- map angles to angles of the same measure.

However, dilations DO NOT, in general, preserve distances. The only dilation with the center at the origin that preserves distances is the identity transformation $(x, y) \rightarrow (x, y)$, which has scale factor $s = 1$.

The Angle-Angle Criterion for Similarity of Triangles

A-A criterion: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

In the figures to the right, if the sums of the angles in both triangles are to be 180°, then angles *c* and *f* must have the same measure. Therefore, the two triangles must be similar.

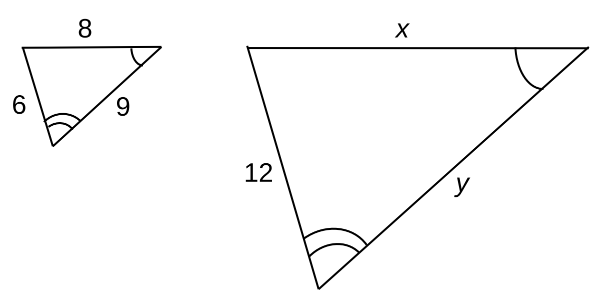


Finding Side Lengths of Similar Triangles

If the two triangles below are similar, there are two basic ways to set up proportions to find missing side lengths.

Method 1: Establish values of ratios of corresponding segments **between** the two figures.

$$\frac{6}{12} = \frac{8}{x} \rightarrow x = 16 \quad \text{and} \quad \frac{6}{12} = \frac{9}{y} \rightarrow y = 18$$



Method 2: Establish values of ratios of corresponding segments **within** the two figures.

$$\frac{8}{6} = \frac{x}{12} \rightarrow x = 16 \quad \text{and} \quad \frac{9}{6} = \frac{y}{12} \rightarrow y = 18$$