# *MATHLINKS*: CORE 2ND ED (GRADES 6-8)
# COMMENTARY ON THE MATH BACKGROUND NOTES

By Ted Gamelin
Professor Emeritus, UCLA Department of Mathematics

The course materials include a set of about seventy different Math Background Notes that provide various types of background materials with different aims. The great majority are written to give the teacher greater insight into the materials. The explanations go beyond the course material in one way or another. They were prepared by a research mathematician with experience teaching teacher preparation courses.

As a whole, the idea of the Math Background Notes is to present a picture to the teacher of mathematics as a discipline based on precise definitions followed by logical reasoning based on the definitions. A recurring theme is the importance of definition, the variation in definitions both historically and in contemporary school materials, and the consistent development of mathematics starting with the definitions. They provide historical development of some of the most important ideas of mathematics, discussions of the various ways the fundamental concepts have been defined, and in some cases the actual derivations of some of the familiar rules of algebra based on the definitions and logical reasoning. They also treat such topics as the use of informal math vocabulary and "slogan forms" of the most important theorems, which the teacher should be aware of and can use as a helpful tool even though the statements are technically inadequate. One of the Math Background Notes provides applications to the science of geodesy that shows how messy mathematics might become when it is actually applied.

## History of Mathematics

Teachers understand their course content better when they have an understanding of its historical development. The Math Background Notes covering history provide insight into where and how the mathematics originated. The discussions provide jumping-off points for teachers expanding their knowledge and understanding of the concepts they are covering in class. Examples with references to history include:

7-1* Origins of Probability: Relates the birth of one of the most important ideas in mathematics, namely, the idea of "mathematical expectation" in probability theory. The theory of probability was relatively late in being placed on firm mathematical grounds, but now the ideas of probability have become of supreme importance in many aspects of our daily live, through algorithms for treating data in large data sets to strategies for investing in the market for maximum financial gain. The field of statistics rests firmly on probability theory. Knowledge of how the field of probability originated strengthens our understanding of probability and statistics.

*Grade and Unit indicator.  As an example, 7-1 represents Grade 7, Unit 1

7-8, 8-1 Why Does a Circle Have 360 Degrees: This is a historical piece that traces the sexagesimal nature of many of our measurements back to the Sumerians and gives some plausible explanations of how such systems might have arisen. At the same time, this provides an explanation of the importance of mathematics for the most advanced early civilizations, to keep records of warehouse supplies and costs of trade items. This short Math Note has the effect of opening a window for interested teachers, or students, to read about ancient civilizations and their mathematical accomplishments.

7-9 The Number Pi: Provides a history of progress by mathematicians in understanding the number pi, which is perhaps the most famous number in mathematics. The basic properties of pi have been difficult to suss out.

8-1 Angle Measurement: Explains how the Greek mathematicians viewed angles. Their definition of an angle was much more limited than the treatments of angles in contemporary textbooks. This Math Note provides a peek into Euclid's Elements, which is the most influential mathematics textbook ever written. Even into the 1900's, it was used in many schools as the standard geometry textbook and had more copies in print than any other book except for the Bible.

8-2 The Parallel Postulate: Provides historical insight into the discovery of non-Euclidean geometries, which is an important theme in eighth grade.

8-2 President Garfield's Proof of the Pythagorean Theorem: This is an interesting mathematical curiosity. Though it was not a significant mathematical accomplishment, it does provide a snapshot of mathematics in the last half of the nineteenth century. Early in his career Garfield was a math teacher and a school administrator, with a firm grounding in Latin and Greek.

8-6, Lines of Best Fit: This Math Note talks in general terms about the problem of finding a curve of best fit for a data set. The appropriate curve for the problem might be a straight line, or a quadratic, or an exponential curve, and then the problem at hand is to determine the parameters of such a curve for which the curve comes closest to the data in some sense. This method is referred to as "regression analysis". It goes back to the work of Francis Galton, who established that the heights of offspring tend to regress to the mean. The work of Gauss using the method of least squares to predict the path of the asteroid Ceres is also mentioned as an important contribution to regression analysis.

8-9 Euclidean Geometry is not the Only Geometry: A geometry provides us with notions of length and angle. This Math Note describes these notions for the three two-dimensional geometries of most interest to mathematicians, which are Euclidean geometry, spherical geometry, and hyperbolic geometry. It describes the contribution of Euclid toward developing Euclidean geometry.

8-9, 8-10 Approaching Congruence and Similarity Through Transformational Geometry: Provides insight and perspective into a fundamental shift in school mathematics from the approach laid out in Euclid's Elements to an approach based on certain transformations of the plane, namely, translations, rotations, reflections, and also dilations. This shift took place in American school mathematics primarily toward the end of the 1900's. An important turning point in this shift was a talk that a German mathematician, Felix Klein, gave at the University of Erlangen. He proposed what became known as the "Erlanger Programm", to interpret the basic ideas of geometry in terms of classes of transformations of the ambient space. In addition to being a leading research mathematician, Klein was the leading figure in mathematics teacher education at the end of the nineteenth century. He is of particular interest to Americans as several of his students immigrated to the United States and played an important role in the development of mathematics in this country, particularly the University of Chicago, which he visited.

## Connections to the Outside World

6-3, 7-3 Ratios Are Everywhere: Gives a load of examples of ratios in ordinary life.

6-10 What Is Sea Level?: Discusses the notion of sea level in the real world. This Math Note is an introduction to the science of geodesy. It is important because it illustrates the complications that occur when mathematics is applied to a real world situation. This particular application is important because of the increasing attention being focused on the rise of sea level.

## Evolution and Multiplicity of Definitions

The house of mathematics is built on a foundation of definitions. To understand the definitions, the knowledge of the teacher should extend beyond the definition in the textbook. The teacher should realize that there may be variants of the definition in other contemporary materials that the teacher may consult for information, which may not be equivalent to the textbook definition. The teacher benefits from knowing something about the historical development of the definition.

6-1, 7-10, "The Quartile" or "In the Quartile": Focusses on two different concepts that are both referred to as "quartile". If the data are placed in order on a number line and divided into four equal parts, the three dividing points are referred to as the quartiles. The four equal parts are also referred to as quartiles. Thus a quartile can mean either a point or a subset of the data set. Which meaning is appropriate must be determined from context. In applications, a quartile usually refers to a data subset. In economics, for instance, it is quite common to divide the population of the United States into four quartiles based on age, or upon family income. In addition to quartiles, data sets are often broken into quintiles or deciles for statistical studies.

6-2 Why Is 1 neither Prime nor Composite?: Traces the status of the number "1". The definitions of prime number and composite number have evolved to make statements of many theorems simpler. This saves a lot of ink.

6-4 Definition of Division: Emphasizes the definition of division of numbers, because so much of the structure of the house of mathematics rests on it.

6-9, 7-8, 7-9 Different Definitions of Quadrilaterals: Provides a warning to teachers about the various different definitions of “trapezoid”.

6-9, 7-9 Base, Altitude, and Height: It is important to realize that the concept of “base” is flexible. For purposes of applying a theorem, one can specify which side is the base.

6-10 Two Possible Definitions of Absolute Value:  Reminds teachers that for a proof based on absolute value, they may use either of two definitions of absolute value. The definitions are equivalent.

7-10 Mean Absolute Deviation: Explains to the teacher why this definition of a measure of spread is introduced in seventh grade, while the more common notion of standard deviation is not introduced until eight grade.

8-4 Different Definitions of Function: The notion of “function” is one of the most important concepts in mathematics. In the middle of the 1900’s the proponents of the new math proposed replacing the input-output definition with a definition based on ordered pairs, essentially the graph of function. The leading calculus textbook (Thomas) briefly replaced the input-out definition with the ordered pair definition, only to find that sales plummeted. Students could not easily understand the new definition, nor could some of the teachers. This Math Note provides a window into that discussion and an opportunity for teachers to broaden their understanding of function.

8-5 Growing the Formal Definition of Slope: Discusses the evolving definitions of slope culminating in the definition given in calculus courses.

8-9 Functions in Mathematics: This Math Note serves as a key to identifying the notion of function when it appears in one of its many guises. It describes how the notion of function varies both according to grade level in the curriculum and according to the particular area of mathematics or science where it is being applied.

## Clarifying Definitions

The following Math Background Notes are devoted to simply stating and clarifying a definition.

6-1 Dependent and Independent Events: Clarifies the notions of dependence and independence through discussion of examples.

6-3 Ratios and Rates: Explains the need for common definitions, and gives the Common Core definitions for ratio, value of a ratio, and unit rate.

6-5, 7-2 ”Percent” and “Percent of”: Students have a difficult time handling the usual definition of percent, though through practice they eventually become proficient in calculating and working with percents. They can find a percent of something, but they cannot say what a percent is. This Math Note clarifies the main issue, at least from a mathematical point of view. The point is that a “percent” is simply a number with a unit attached. The unit is denoted by %, and 1% is simply the number 0.01.  “Percent of” something can be defined simply as the number times the something. For instance 5%

of something is simply the number 5% = 0.05 times the something. Something similar occurs in finance when the yields of bonds are described in basis points. The basis point is a number unit defined by 1 bp = 0.01% = 0.0001.

6-7 Independent and Dependent Variables: The determination of which variables are independent and which dependent is not inherent. It depends on the specific problem being addressed. To a certain extent, the researcher has flexibility in specifying the independent and the dependent variables.

6-9, 7-9 Polygons: Illustrates basic definitions pictorially. It includes a statement of the Jordan Curve Theorem for simple closed polygonal curves in the plane, asserting that any such curve divides the plane into two connected components. This statement may seem obvious, but in fact a proof requires a fair amount of work. I once assigned the proof as a “problem of the week” to an undergraduate class in a teacher education program, to see how the students would do. No one gave a correct proof, even after doing web searches. I did learn a lot about the students.

6-9, 7-8, 7-9 Polyhedra: Clarifies basic definitions, with figures.

7-2 Simple Versus Compound Interest: Clarifies the definitions, with examples.

7-3 Ratio, Rate, Unit Rate, and Value: Clarifies the definitions.

7-3 Geometric Interpretation of Equivalent Ratios: Interprets the definition of equivalent ratios by means of straight-line graphs.

7-3 Equivalent Ratios and Proportional Relationship: Clarifies the definition of proportional relationship.

8-2, 8-3 The Principal Square Root: Basic definition without explanation.

8-6 Numerical and Categorical Data Sets: Describes various kinds of data sets that arise, with examples. The Math Note also discusses clumping data into classes.

8-6 Association and Causation: This Math Note discusses the notion of association of two variables, and it underscores the important distinction between association and causation. It mentions linear associations and correlation coefficients. It provides several examples relevant to modern life.

8-6 Categorical Variables and Frequencies: Provides formal definitions of a categorical variable and the frequency of a categorical variable, along with examples.

8-6 Bivariate Data and Two-Way Tables: Discusses by means of an example the representation of a bivariate data set in two-way frequency table and in two kinds of two-way relative frequency tables. Contingency tables are mentioned.

8-10 Congruence and Similarity are Equivalence Relations: The main focus of this Math Note is the definition of an equivalence relation. The Math Note provides two examples of equivalence relations, namely, congruence of geometric figures, and similarity of geometric figures. It points out the properties of rigid transformations that imply that congruence is an equivalence relation, and the properties of similarity transformations that imply that similarity is an equivalence relation.

Theorems, Proofs, and Logical Reasoning

6-4 The Fundamental Theorem of Arithmetic: This Math Note states the Fundamental Theorem of Arithmetic and gives an example. It points out a fact, easy to gloss over, that the Fundamental Theorem of Arithmetic actually consists of two theorems. One theorem states that any number $n \geq 2$ is a product of prime numbers. The second theorem states that the representation is essentially unique, that is, any one representation can be obtained from any other by rearranging the order of the primes.

6-4 Rules for Division of Fractions: Demonstrates how the rules for manipulating fractions that students should become familiar with can be derived from the definition of division. What is important here is not the specific derivations of the various rules. To a certain extent, to go through these derivations carefully would be mind-deadening. What is important is that the rules can be derived from the definition of division, somebody else has done it in careful detail, and if you want to you can check that the derivation is correct.

6-9 Alternate Methods for Determining Area Formulas by Composing and Decomposing Figures: Describes several cut-up arguments for deriving the area formulas for planar figures, one for a triangle and two for trapezoids. These augment the cut-up arguments given in the text. They serve to familiarize students with the various polygonal figures.

7-3 Reasoning and Proof: Why Cross-Multiplication Works: Provides a two-column proof of the cross-multiplication rule based on properties of arithmetic.

7-3 How Much Detail Is Needed in a Proof?: This is a commentary on the two-column proof of the cross-multiplication rule. It points out that some piddling details were omitted in order to make the proof more transparent.

7-5 Why Does (-1)(-1) = 1?: Provides a two-column proof, based on properties of arithmetic and equality.

7-5 Derivation of Sign Rules for Multiplication: Provides proofs of the sign rules.

7-5 Why Is Division by Zero Undefined?: Provides four different ways of understanding why we cannot divide by zero.

7-9, When Is a Proof a Proof?: Addresses the problem of how much detail should be written in a proof for the proof to be considered correct. Too little detail might leave gaps, and too much detail might obscure the main points. This Math Note quotes Hyman Bass, a distinguished research mathematician who teamed up with a leading math educator Deborah Loewenberg Ball to form an awesome team, perhaps the leading math educators of the first part of this century.

8-1 Angle Sums of Triangles: Demonstrates how the postulates and previously established facts are used to prove two theorems about the angles of a triangle. This Math Note demonstrates how a two-column format is used for proofs.

8-2 Statements of the Pythagorean Theorem: Discusses the statement and various abbreviated statements, including “slogan forms” of the Pythagorean Theorem. Slogan forms of theorems can be useful as mnemonic devices, but it should be realized that they are not formally correct.

8-2 The Converse of a Theorem: Discusses statements of converses of theorems with several examples.

8-4 The Obvious Rule May Not Be the Only Rule: Points out that for a finite sequence of integers there are many recursion formulae that give rise to the sequence.

8-5 Why Can’t You Divide By Zero?: Provides three mathematical explanations why we cannot divide by zero.

8-8 Conjecture Versus Proof, Number Tricks: Discusses inductive reasoning, deductive reasoning, conjecture, and generalization. Illustrates these terms through a standard “number trick”.

8-9 Informal Math Vocabulary: Gives examples of informal math terms and their formal counterparts, as “flip” versus “reflection”. Informal math terms can be useful pedagogical tools, as they are easier to engrave in the memory.

8-10 The Angle-Angle (A-A) Criterion for Similarity of Triangles: Proves that if two angles of one triangle are congruent respectively to two angles of another, then the triangles are similar.

8-1- Similarity and Slope: Uses the A-A Criterion for similarity of triangles to prove that the definition of slope of a line does not depend on the right triangle used to determine the quotient of rise over run of the line.

<u>Operations and Variables</u>

6-6, 7-5 Order of Operations: Refers to a convention for evaluating algebraic expressions. The Math Note shows how important it is to adopt a convention.

6-10, 7-4 Interpreting the Minus Sign: Discusses three different interpretations of the minus sign.

6-6 Variables in Algebra: This is a bare-bones summary of a doctoral dissertation analyzing the various uses of variables in mathematics.

<u>Real Numbers and Decimal Expansions</u>

7-1 A Clever Trick: This Math Note gives the procedure for determining the rational number corresponding to a given repeating decimal expansion.

8-2 Decimal Expansions of Rational Numbers: This pithy Math Note has discussion with proofs of three things to know about decimal expansions and rational numbers, that every rational number has a repeating decimal expansion, that every repeating decimal expansion is the decimal expansion of a rational number, and that the rational numbers with terminating decimal expansions are those of the form m/n where the only possible prime factors of n are 2 and 5.

8-2 Subsets of the Real Numbers: This crucial Math Note has what can be regarded as the definition of real numbers as decimal expansions. The rational numbers are the repeating or terminating decimal expansions, where each terminating decimal expansion is regarded as the same national number as a decimal expansion terminating in all 9's. The irrational numbers are the decimal expansions that are neither repeating nor terminating. We may regard the real line as houses on a long street. Every house has an address, which is a decimal expansion, except that certain houses have two address, corresponding to a front door and a side door. What we have described above is one of the methods for constructing the real numbers from the rational numbers. There are two other methods from the nineteenth century for constructing the real numbers, the method of Dedekind cuts, and the method of metric completion. The other two methods have the advantage that the arithmetical operations are easier to extend and yield the usual results. On the other hand, if an eighth-grader asks you what really are real numbers, you are best off explaining them as decimal expansions.

8-2 Place Value Names: This Math Note has a diagram giving the place value names of the digits in a decimal expansion.

<u>Miscellaneous</u>

6-1 The Tale is in the Tail: Plants a picture to illustrate vocabulary, along with a catchy phrase. A picture is worth a thousand words.

7-1 What Is Randomness?; Gives an informal definition of a random events, along with examples. Discusses random sequences of integers. P
oints out some common misperceptions about truly random sequences.

7-9 Notation Clarification – "B" vs. "b": In mathematics, the selection of appropriate notation is extremely important, both for writing and for teaching. Good notation facilitates understanding. Inconsistent notation torpedoes understanding. Bad notation addles the brain. This Math Note gives a nod to the importance of notation by discussing the choices made for measures of length and area of geometric figures, specifically in the formulas A = bh and V = Bh.

7-10 Sampling: One of the most important applications of statistics is to draw conclusions about a population by selecting a representative sample of the population and gathering information from that sample. From the results for the sample, the mathematics of probability distributions allows us to draw certain probabilistic conclusions about the entire population. This Math Note gives an overview of sampling, including various strategies such as stratified sampling and clumped sampling. Behind these strategies is usually the goal of obtaining maximal accuracy at minimal cost.

8-2 Approximating Square Roots by Linear Interpolation: This Math Note defines what is meant by linear interpolation, and it uses the method to approximate the square root of 27 by interpolating linearly between the points (25,5) and (36,6) of the graph of the square root function. It shows how proportional reasoning can be used to shorten the calculation.

8-3 Simplifying Radical Expressions: Discusses what it means to simplify a radical expression. There is no fixed notion of what the simplified form of a radical expression should be, but usefulness and attractiveness both play a role.

8-9 Basic Properties Shared by Translations, Rotations, and Reflections: This Math Note lists four properties shared by these rigid transformations, namely, they preserve distances, they preserve angular measure, they preserve parallelism, and they map lines to lines. The Common Core State Standards require middle grade students to become familiar with these properties, but not that they verify these properties formally.

8-10 Properties of Dilations: This Math Note points out that dilations share some of the properties of rigid motions. They preserve angular measure, they preserve parallelism, and they map lines to lines. However, they do not preserve distances.