STUDENT RESOURCES

Word or Phrase	Definition
complex fraction	A complex fraction is a fraction whose numerator or denominator is a fraction. Two complex fractions are $\frac{\frac{4}{5}}{\frac{1}{2}}$ and $\frac{\frac{1}{5}}{3}$.
constant of proportionality	See proportional.
dependent variable	A dependent variable is a variable whose value is determined by the values of the independent variables. See independent variable.
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8+x}{10}$, and $4v - w$.
equivalent ratios	Two ratios are equivalent ratios if each number in one ratio is obtained by multiplying the corresponding numbers in the other ratio by the same positive number. 5 : 3 and 20 : 12 are equivalent ratios because both numbers in the ratio 5 : 3 are multiplied by 4 to get to the ratio 20 : 12.
independent variable	An independent variable is a variable whose value may be specified. Once specified, the values of the independent variables determine the values of the dependent variables. For the equation $y = 3x$, y is the dependent variable and x is the independent variable. We may assign a value to x. The value assigned to x determines the value of y.

<table>
<tr><th>Word or Phrase</th><th>Definition</th></tr>
<tr><td>input-output rule</td><td>An input-output rule for a sequence of values is a rule that establishes explicitly an output value for each given input value.
<table><tr><td>input value (x)</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>x</td></tr><tr><td>output value (y)</td><td>1.5</td><td>3</td><td>4.5</td><td>6</td><td>7.5</td><td>$1.5x$</td></tr></table>
In the table above, the input-output rule could be $y = 1.5x$. In other words, to get the output value, multiply the input value by 1.5. If $x = 100$, then $y = 1.5(100) = 150$.</td></tr>
<tr><td>proportional</td><td>Two variables are proportional if the values of one are the same constant multiple of the corresponding values of the other. The variables are said to be in a proportional relationship, and the constant is referred to as the constant of proportionality.

If Wrigley eats 3 cups of kibble each day, then the number of cups of kibble is proportional to the number of days. If x is the number of days, and y is the number of cups of kibble, then $y = 3x$. The constant of proportionality is 3.</td></tr>
<tr><td>proportional relationship</td><td>See proportional.</td></tr>
<tr><td>ratio</td><td>A ratio is a pair of positive numbers in a specific order. The ratio of a to b is denoted by $a : b$ (read "a to b," or "a for every b").

The ratio of 3 to 2 is denoted by 3 : 2. The ratio of dogs to cats is 3 to 2. There are 3 cups of water for every 2 cups of juice. The fraction $\frac{3}{2}$ does not represent this ratio, but it does represent the ratio's value (or the unit rate).</td></tr>
<tr><td>unit price</td><td>A unit price is a price for one unit of measure.</td></tr>
<tr><td>unit rate</td><td>The unit rate associated with a ratio $a : b$ of two quantities a and b, $b \neq 0$, is the value $\frac{a}{b}$, to which units may be attached.

The ratio of 40 miles each 5 hours has unit rate of 8 miles per hour.</td></tr>
<tr><td>value of a ratio</td><td>See unit rate.</td></tr>
<tr><td>variable</td><td>A variable is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to quantities that vary in a relationship (as in a formula or an input-output rule). They may refer to unknown quantities in expressions, equations or inequalities. Finally, they may be used to generalize rules of arithmetic.

In the equation $d = rt$, the quantities d, r, and t are variables.
In the equation $2x = 10$, the variable x may be referred to as the unknown.
The equation $a + b = b + a$ generalizes the commutative property of addition for all numbers a and b.</td></tr>
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Testing for a Proportional Relationship

Here are three ways to test if two variables are in a proportional relationship:

- The values of the ratios (unit rates or unit prices) created by data pairs are equivalent.
- An equation in the form $y = kx$ fits all corresponding data pairs.
- Graphed data pairs fall on a line through the origin (0, 0).

Note that this example does **not** represent a proportional relationship. Alexa buys tickets when she goes to the amusement park. This chart shows the costs for different quantities of tickets.

# of tickets	10	20	25	50	100
total cost	$40	$60	$75	$125	$200
cost per ticket	$4	$3	$3	$2.50	$2

Since the costs per ticket (unit prices) are not the same, ticket purchasing at this amusement park does **not** represent a proportional relationship.

This example **does** represent a proportional relationship. Antonio kept track of the number of miles he traveled each time he filled his tank with gas. Here is some data.

number of miles	100	200	175	300
number of gallons	4	8	7	12
miles per gallon	25	25	25	25

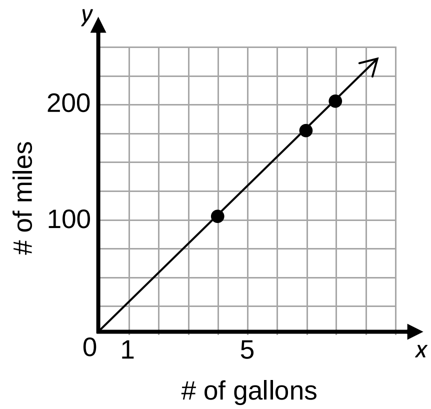
Since the miles per gallon (unit rates) created by the data pairs is the same, this situation represents quantities in a proportional relationship.

Furthermore,

Let x = the number of gallons
Let y = the number of miles

The data fits the equation $y = 25x$ (an equation in the form $y = kx$), which is an equation that represents a proportional relationship.

Finally, if the points for (gallons, miles) are graphed, they will fall on a line through the origin (0,0).



Multiple Representations and Proportional Relationships

Suppose 4 balloons cost $6.00 and each balloon is the same price. Here are some strategies for representing this proportional relationship.

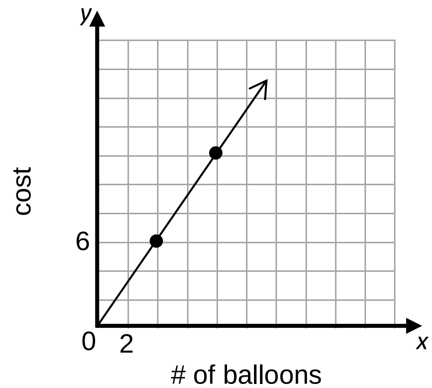
Strategy 1: Tables

Create a table to calculate unit rates. If the unit rates are the same, the variables are in a proportional relationship.

Number of Balloons	Cost	Unit Price
4	$6.00	$1.50
2	$3.00	$1.50
1	$1.50	$1.50
8	$12.00	$1.50

Strategy 2: Graphs

A straight line through the origin indicates quantities in a proportional relationship.



Strategy 3: Equations

An equation of the form $y = kx$ indicates quantities in a proportional relationship. In this case,

y = cost in dollars
x = number of balloons
k = cost per balloon (unit price)

To determine the unit price, create a ratio whose value is: $\frac{6 \text{ dollars}}{4 \text{ balloons}} = 1.50 \frac{\text{dollars}}{\text{balloons}}$

Therefore, k = $1.50 per balloon, and $y = 1.50x$.

This equation expresses the output as a constant multiple of the input, showing that the relationship is proportional.

Sense-Making Strategies to Solve Proportional Reasoning Problems

How much will 5 pencils cost if 8 pencils cost $4.40?

Strategy 1: Use a "halving" strategy

If 8 pencils cost $4.40, then
4 pencils cost $2.20,
2 pencils cost $1.10, and
1 pencil costs $0.55.

Therefore, 5 pencils cost

$0.55 + $2.20 = $2.75.

Strategy 2: Find unit prices

First, find the cost of one pencil.

$$\frac{\$4.40}{8} = \$0.55$$

Then, multiply by 5 to find the cost of 5 pencils,

($0.55)(5) = $2.75.

Sammy can crawl 12 feet in 3 seconds. At this rate, how far can Sammy crawl in $1\frac{1}{2}$ minutes?

Strategy 1: Make a table

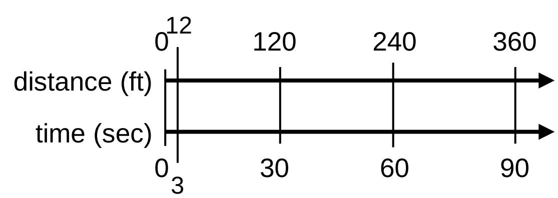
Distance	Time
12 ft	3 seconds
4 ft	1 second
240 ft	60 sec = 1 min
120 ft	30 sec = $\frac{1}{2}$ min
360 ft	90 sec = $1\frac{1}{2}$ min

Sammy can crawl 360 feet in $1\frac{1}{2}$ minutes.

Strategy 2: Make a Double Number Line

12 feet in 3 seconds is equivalent to
120 feet in 30 seconds

$1\frac{1}{2}$ minutes = 90 seconds.

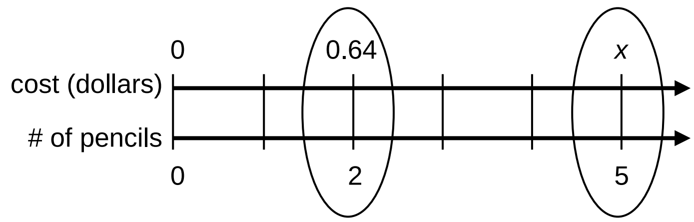


Sammy can crawl 360 feet in $1\frac{1}{2}$ minutes.

Writing Equations Based on Rates

Here are some ways to set up an equation to solve a rate problem. An equation in the form $\frac{a}{b} = \frac{c}{d}$ is commonly referred to as a "proportion." Double number lines help make sense of this process. (See boxes on the next page for equation solving strategies.)

If 2 pencils cost $0.64, how much will 5 pencils cost?



Strategy 1: Compare rates ("between" two different units)

Create two rates from ratios that compare dollars to pencils. Equate expressions and solve for x.

$$\frac{x}{5} = \frac{0.64}{2}$$

x = 1.60 dollars for 5 pencils.

Note: The equation $\frac{5}{x} = \frac{2}{0.64}$ is another valid "between" equation for this problem.

Strategy 2: Compare like units ('within" the same units)

Create one rate based on corresponding cost ratios and another rate based on the corresponding numbers of pencils ratios. Then, equate expressions and solve for x.

$$\frac{\text{cost}_{\text{case 1}}}{\text{cost}_{\text{case 2}}} = \frac{0.64}{x} \qquad \frac{\text{pencils}_{\text{case 1}}}{\text{pencils}_{\text{case 2}}} = \frac{2}{5}$$

$$\frac{0.64}{x} = \frac{2}{5}$$

x = 1.60 dollars for 5 pencils.

Note: The equation $\frac{x}{0.64} = \frac{5}{2}$ is another valid "within" equation for this problem.

Some Properties Relevant to Solving Equations

Here are some important properties of arithmetic and equality related to solving equations.

- The <u>multiplication property of equality</u> states that equals multiplied by equals are equal. Thus, if $a = b$ and $c = d$, then $ac = bd$.

  Example: If $1 + 2 = 3$ and $5 = 9 - 4$, then $(1 + 2)(5) = 3(9 - 4)$.

- The <u>cross-multiplication property for equations</u> states that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ ($b \neq 0$, $d \neq 0$).

  This can be remembered with the diagram: $\frac{a}{b} \times \frac{c}{d}$.

  Example: If $\frac{5}{7} = \frac{12}{x}$, then $5 \bullet x = 7 \bullet 12$.
  To see that this property is reasonable, try simple numbers:
  If $\frac{3}{4} = \frac{6}{8}$, then $3 \bullet 8 = 4 \bullet 6$.

Applying Properties to Solve Proportion Equations

Strategy 1:
Multiplication Property of Equality

Solve for x:

$$\frac{x}{12} = \frac{3}{8}$$

Multiplication Property of Equality

$$(8 \bullet 12) \bullet \frac{x}{12} = \frac{3}{8} \bullet (8 \bullet 12)$$

$$8x = 36$$

$$x = \frac{36}{8}$$

$$x = 4\frac{1}{2}$$

Strategy 2:
Cross-Multiplication Property

Solve for x:

$$\frac{x}{12} = \frac{3}{8}$$

Cross-multiplication property

$$8 \bullet x = (3 \bullet 12)$$

$$8x = 36$$

$$x = \frac{36}{8}$$

$$x = 4\frac{1}{2}$$

Simplifying Complex Fractions

Strategy 1: A complex fraction can be written as a division problem.

Example: $\dfrac{\frac{1}{4}}{\frac{3}{8}} = \dfrac{1}{4} \div \dfrac{3}{8} = \dfrac{1}{4} \bullet \dfrac{8}{3} = \dfrac{8}{12} = \dfrac{2}{3}$

Strategy 2: A complex fraction can be multiplied by a form of the “big one” to create a denominator equal to one. Multiply the numerator and denominator each by the reciprocal of the denominator (in this case since the reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$). This process leaves a multiplication problem to compute.

Example: $\dfrac{\frac{1}{4}}{\frac{3}{8}} \bullet \dfrac{\frac{8}{3}}{\frac{8}{3}} = \dfrac{\frac{1 \bullet 8}{4 \bullet 3}}{\frac{3 \bullet 8}{8 \bullet 3}} = \dfrac{\frac{8}{12}}{1} = \dfrac{8}{12} = \dfrac{2}{3}$

While Strategy 2 seems to require more steps, this strategy makes more transparent the properties involved in writing the complex fraction in a more usable form.