TEACHER EDITION

MathLinks

GRADE 7

COMPILED MATH BACKGROUND

MATH BACKGROUND

Origins of Probability

An important day in the history of mathematics occurred in 1653 when Blaise Pascal, Antoine Gombaud, and Chevalier de Méré, found themselves sharing a carriage ride with a mutual friend. The Chevalier de Méré was not actually a nobleman – he was a writer who had invented the name “Chevalier de Méré” for a character who espoused his views in the dialogues he wrote. Apparently, his friends started calling him Chevalier de Méré, which he liked, and he eventually adopted the name for himself.

The Chevalier de Méré gambled, and he was very interested in the mathematics connected to gambling, presumably to assure that he had the advantage in any game of chance. As they bumped along in the carriage, he told Pascal about several problems that arise in gambling. This opened up new horizons for Pascal. The enthusiastic Pascal tackled the problems and made some headway. At the time, Pascal was in correspondence with Pierre de Fermat, one of the great mathematicians of the century, so it was natural for Pascal to write to Fermat about these problems. This correspondence between Pascal and Fermat, in the year 1654, laid the foundations of what would become modern probability theory.

Two specific problems that the Chevalier de Méré told Pascal about, and which were under some discussion at the time, had been treated by Cardano some hundred years earlier and subsequently by a number of mathematicians. The first problem ran along the following lines:

A dice thrower bets a sum of money—against anyone who matches the sum—that in 24 throws of a pair of dice he will roll a pair of sixes. De Méré thought that the dice thrower should have an advantage. The intuitive argument for the thrower to have an advantage was that the odds of getting a pair of sixes in one throw is 1 out of 36, so in 24 throws the odds of getting a pair of sixes should be about 24 out of 36, or 2 out of 3. It seemed like a safe bet, but de Méré was losing money. He thought he had discovered a contradiction in mathematics. (It turns out that to have an advantage, the dice thrower should bet that in 26 throws he will get a pair of sixes.)

The second problem concerned how the stakes of a game played in stages should be divided between two gamblers if the game must be terminated before it is over. This problem, called the “problem of points,” was solved by Fermat in his correspondence with Pascal. The solution led to the first explicit reasoning about what today are known as expected values.

Dependent and Independent Events

A bag contains 2 green cubes and 1 brown cube. Two cubes are to be drawn from the bag.

Experiment #1: Suppose the cubes are removed from the bag **without** replacement. The events of "drawing a green cube on the first draw" and "drawing a brown cube on the second draw" are dependent. The likelihood of the second event (of drawing a brown cube on the second draw) is influenced by the occurrence of the first event (that is, by whether or not a green cube was drawn first).

Experiment #2: Now suppose one cube is removed from the bag, the color is noted, and it is replaced. After shaking the bag, another cube is removed from the bag and the color is noted. In this experiment of drawing **with** replacement, the event of drawing a green cube on the first draw is independent of the event of drawing a brown cube on the second draw. The likelihood of drawing a brown cube on the second draw is equal to the likelihood of drawing a brown cube on the first draw. It is not influenced by whether or not a green cube was drawn on the first draw.

What is Randomness?

Events that have no pattern or are unpredictable are referred to as random events. Rolling a 1 on a fair number cube is a random event. Rolling a 1 has probability $\frac{1}{6}$ of occurring on any given roll, but otherwise the occurrence of a 1 is unpredictable, and there is no pattern in the appearance of 1's in a sequence of rolls.

Though the results of a probability experiment (such as rolling a number cube) may be unpredictable in the short term, certain statements can be made about the results of repeated trials in the long term. In the long term, the average number of 1's appearing (the experimental probability of rolling 1) will get closer and closer to $\frac{1}{6}$ as the number of rolls increases.

In fact, it is a remarkable fact that if we repeatedly perform a probability experiment of this sort, the average of the outcomes will "fit" on a normal distribution with a specific mean and standard deviation. This signals the very special role that the normal distribution plays in mathematics and statistics.

If a player with a fair number cube rolls nine 1's in a row, what is the probability that the player will get a 1 on the next roll? Though the answer may seem counterintuitive, the probability of rolling a 1 after having rolled nine consecutive 1's is still $\frac{1}{6}$. The tenth roll is independent of the first nine rolls.

It is virtually impossible for people to discern whether a given sequence of digits, as the results of rolling a number cube, is random or not. However, if we roll a die enough times, seemingly impossible streaks such as 1,1,1,1,1,1 are almost certain to occur. In fact, there are statistical tests to determine whether a given sequence of digits is random, based on the occasional appearance of long strings of the same digit.

Truly random sequences, though, do not appear random to people. Sports fans regularly attribute NBA players' streaks to hot hands, even when their performance is no better than chance. A Spotify developer reported that their random shuffle feature did not feel random to many of their listeners, and subsequently received many complaints. They decided to replace their truly random shuffle algorithm by an algorithm that fits the common intuition of a random selection.

A Clever Trick

Rational numbers have decimal expansions that repeat, either with a repeating pattern of nonzero digits, or with zeros from some point on (terminating decimals). This occurs because the decimal expansion of a quotient of integers obtained by long division eventually repeats, since there are only a finite number of possibilities for the remainder. For example, if the divisor is 7, then the only possibilities for the remainder are 0, 1, 2, 3, 4, 5, 6.

Conversely, every repeating decimal is the decimal expansion of a rational number. There is a "clever trick" for converting repeating decimals to fractions. We illustrate the clever trick by converting the decimal $x = 0.16666\ldots$ to a quotient of integers, following the scheme used in the Student Pages. Notice the unconventional order in writing down the steps. This is done to simplify the arithmetic for the students.

$10x = 1.66666\ldots$ (2) **Let $\boldsymbol{x = 0.16666\ldots}$ (1)** $9x = 1.5$ (3) $x = \frac{1.5}{9} = \frac{15}{90} = \frac{1}{6}$ (4)	Notice that step 2 is above step 1. • The clever trick is to multiply both sides of the equation in step 1 by a power of 10 that will "line up" the repeating portion of the decimal. • Then subtract the expressions in step 1 from step 2. This will make the repeating portion equal zero (step 3). • Finally, solve for x and simplify your result into a quotient of integers (step 4).

The procedure for converting a decimal to a rational number gives a different rational number for each decimal except in the special case that the decimal is terminating or is repeating with all 9's. For instance, it shows that 0.29999…. represents the same rational number as 0.3.

Students are often surprised that 0.9999… is a decimal representation of 1, as can be seen by using the clever trick to convert 0.9999… to a quotient of integers.

$$\begin{aligned} 10x &= 9.999\ldots \\ \underline{\text{Let } x} &\underline{= 0.999\ldots} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

MATH BACKGROUND

Percent AS a Number and Percent OF a Number

We may regard a “percent” as a number, or we may talk about “a percent of a quantity.” These two concepts are related, but they are different, and the difference may lead to confusion. A parallel situation occurs for fractions. We may regard $\frac{3}{4}$ as a number, or we may talk about $\frac{3}{4}$ of something.

Percent as a number

If P is a nonnegative number, then P percent (denoted $P\%$) is P hundredths:

$$P\% = \frac{P}{100}.$$

We may think of a percent as a number expressed in terms of the unit of measure $\% = 1\% = \frac{1}{100}$.

To convert a percent to a number, divide the percent by 100.

Example: Fifteen percent = 15% = $\frac{15}{100}$ = 0.15

To convert a positive number to a percent, multiply the number by 100.

Example: $4 = 4 \times 100\% = 400\%$

Percent of a number

Three fourths of a number is the product of $\frac{3}{4}$ and the number. Proceeding in analogy, we say that P percent of a number is the product of $P\%$ and the number:

$$P\% \text{ of } n = P\% \bullet n = \frac{P}{100} \bullet n$$

Example: 15% of 300 is $15\% \bullet 300 = \frac{15}{100} \bullet 300 = 45$

Simple vs. Compound Interest

While the Common Core State Standards for Grade 7 mention only simple interest, many financial transactions such as savings account deposits at banks involve compound interest. Simple interest rates are used primarily for loans over short time periods, or for personal loans. Compound interest takes into account the time-value of money by adding interest to the principal after each time period, and charging interest on that interest in subsequent time periods.

Example: You make a two-year loan of $600 at an annual interest rate of 5%. How much interest will you receive, and what is the total amount that you will get back after two years?

Simple Interest

Let I = interest
Let P = principal
Let R = rate
Let T = time
Let A = total amount

$I = PRT$ $\qquad A = P + I$
$I = 600(0.05)(2)$ $\qquad A = 600 + 60$
$I = \$60$ $\qquad A = \$660$

Compound Interest

Suppose the interest is compounded annually.

At the end of year 1:

$I = PRT$ $\qquad A\text{ (year 1)} = P + I$
$I = 600(0.05)(1)$ $\qquad A\text{ (year 1)} = 600 + 30$
$I = \$30$ $\qquad A\text{ (year 1)} = \$630$

At the end of year 2:

$I = PRT$ $\qquad A\text{ (year 2)} = P + I$
$I = 630(0.05)(1)$ $\qquad A\text{ (year 2)} = 630 + 31.50$
$I = \$31.50$ $\qquad A\text{ (year 2)} = \$661.50$

Therefore, you earn $60 in interest over two years with a simple annual interest rate of 5%, while you earn $61.50 ($30 + $31.50) in interest over two years with a 5% annual interest rate compounded annually.

MATH BACKGROUND

Ratios are Everywhere

Under every rug there is a ratio.

In mathematics:

- the value of the ratio of the circumference of a circle to its diameter (π)
- the value of the ratio of lengths of corresponding sides of similar triangles
- the value of the ratios of side lengths of right triangles (trigonometric ratios)
- the value of the ratio of the “increase in the y-variable” to the “increase in the x-variable” (slope of a line)
- conversion rates, such as feet to meters or minutes to hours

In the environment:

- comparisons, such as nineteen out of twenty glaciers are receding
- the ratio of males to females in China (They have a big population problem.)
- the ratio of rabbits to coyotes on Whidby Island, Washington (The rabbit population is exploding! Where are the coyotes?)
- the ratio of electric vehicles to gas-powered vehicles, and of hybrids to gas-powered vehicles on the road
- the ratio of magnitude 3 tornados to magnitude 2 tornados in the United States (There seem to be more tornados of lesser magnitude, and they seem to be moving to the southeast.)

In daily activities:

- two cups water for every cup oatmeal (recipe)
- a dozen almonds per serving
- thirty miles per hour (a speed limit)
- twenty-seven miles per gallon (fuel consumption)

In pricing:

- cheese at $5 per pound
- farmland at $8,000 per acre

In sports and exercise:

- odds of Boston winning the World Series
- calories burned in fifteen minutes jogging

Whenever we refer to percentages, we are using ratios. The battery life of our electronic device, the sales tax on our pizza, and the discount on sale items are given as a percentage**.**

Ratio, Rate, Unit Rate, and Value

The words “ratio” and “rate” have various shades of meaning in common language. The definitions in school mathematics textbooks vary. The Common Core State Standards for Mathematics (CCSS-M) and Progressions prescribe a formal definition of “ratio,” and at least implicitly a definition of “unit rate.” On the other hand, “rate” is treated as a term in common language. No formal definition of “rate” appears in the documents.

- A ratio is a pair of positive numbers in a specific order. The ratio of a to b is denoted by $a : b$ (read “a to b,” or “a for every b”).

  Examples of ratios: $3 : 2$, $\frac{4}{5} : 2$, $3.14 : 10$.

  These are NOT ratios: $0 : 0$, $2 : -3$.

- Unit rate associated with a ratio: Suppose $a : b$ is a ratio, and $b \neq 0$. The unit rate associated to $a : b$ is the number $a \div b$, which may have units attached to it. If a and b have units attached to them, say “a-units” and “b-units,” the appropriate unit of measure for the unit rate is “a-units per b-unit.”

  Example: The ratio “400 miles every 8 hours” has unit rate “50 miles per hour.” There is a convenient calculation device that leads to the unit for the unit rate:

  $$\frac{400 \text{ miles}}{8 \text{ hours}} = \frac{400}{8} \frac{\text{miles}}{\text{hours}} = 50 \frac{\text{miles}}{\text{hours}} = 50 \text{ miles per hour.}$$

- Value of a ratio: The value of a ratio $a : b$, $b \neq 0$, is the quotient number $a \div b$.

  Example: The value of the ratio $6 : 3$ is $6 \div 3 = 2$. The value of the ratio $7 : 2$ is 3.5.
  The value of the ratio “400 miles every 8 hours” is $\frac{400}{8} = 50$.

Both terms “value” and “unit rate” are based on the same numerical value, the quotient number $a \div b$. The difference between the terms is that *all* ratios $a : b$ with $b \neq 0$ have a value, whereas we generally talk about unit rates only for ratios that have units attached to them. In the latter case, the unit rate is equal to the value of the ratio with “something per something” attached.

Geometric Interpretation of Equivalent Ratios

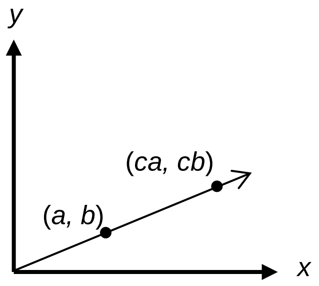
Two ratios are <u>equivalent</u> if each number in one ratio is a multiple of the corresponding number in the other ratio by the same positive number. Thus the ratio $a : b$ is equivalent to the ratio $ca : cb$ for all numbers $c > 0$.

When $b \neq 0$, the <u>value of a ratio</u> $a : b$ is the quotient number $a \div b$. We extend the definition of value to ratios $a : b$ with $b = 0$ by declaring that the value of the ratio $a : 0$ is $+\infty$. This is analogous to thinking of a vertical line in the plane as having slope $+\infty$. Though $+\infty$ is not a number, it is a perfectly legitimate value for a function.

Now that we have extended the definition of value of a ratio to cover the case when $b = 0$, we can give a simple geometrical characterization of equivalent ratios in terms of rays in the plane.

Each ratio $a : b$ determines a point (a, b) in the first quadrant of the coordinate plane. This correspondence $a : b \rightarrow (a, b)$ maps ratios to the first quadrant, including the positive x-axis and y-axis but omitting the origin. Ratios $0 : b$ with value 0 are mapped to points $(0, b)$ on the positive x-axis, and ratios $a : 0$ with value $+\infty$ are mapped to points $(a, 0)$ on the positive y-axis. Under this correspondence, the ratios $ca : cb$ equivalent to $a : b$ correspond to the points (ca, cb) on the ray through (a, b) emanating from the origin. In fact, if we assign a slope of $+\infty$ to a vertical line, then the following statements are valid for all ratios:

- The ratios equivalent to $a : b$ correspond to the ray (half-line) issuing from the origin through (a, b).
- The slope of the ray through (a, b) is the value of the ratio $a : b$.
- Two ratios are equivalent if, and only if, they have the same value.



Equivalent Ratios and Proportional Relationships

Two positive variables x and y are in a <u>proportional relationship</u> if the values of y are the same constant multiple of the values of x, that is, $y = cx$ for some constant c. The constant c is the <u>constant of proportionality</u>. The graph of the pairs of values (x,y) lie on the ray of slope c emanating from the origin.
If x and y are in a proportional relationship, then the ratios $y : x$ of the values of y to the corresponding values of x all have the same value c, since $c = \frac{y}{x}$. Thus the ratios $y : x$ are all equivalent.

Conversely, if the ratios $y : x$ of the values of y to the corresponding values of x are all equivalent, and c is the common value of the ratios, then x and y are in a proportional relationship , and c is the constant of proportionality.

Reasoning and Proof: Why Cross-Multiplication Works for Equations of the Form $\frac{a}{b}=\frac{c}{d}$

Equations in the form $\frac{a}{b}=\frac{c}{d}$ are commonly referred to as proportions.

Prove: If $\frac{a}{b}=\frac{c}{d}$, then $ad = bc$ (assuming $b \neq 0$, $d \neq 0$).

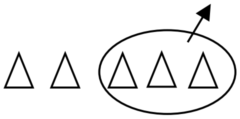
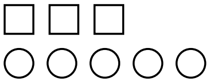
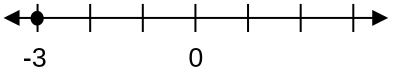
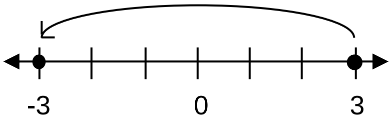
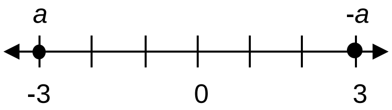
Statement:	**Reason:**
$\frac{a}{b}=\frac{c}{d}$	given
$a \bullet \frac{1}{b} = c \bullet \frac{1}{d}$	definition of division
$(b \bullet d) \bullet \left(a \bullet \frac{1}{b}\right) = (b \bullet d) \bullet \left(c \bullet \frac{1}{d}\right)$	multiplication property of equality
$(a \bullet d) \bullet \left(b \bullet \frac{1}{b}\right) = (b \bullet c) \bullet \left(d \bullet \frac{1}{d}\right)$	commutative and associative properties of multiplication
$a \bullet d \bullet 1 = b \bullet c \bullet 1$	multiplicative inverse property
$a \bullet d = b \bullet c$	multiplicative identity property

How Much Detail Is Needed in a Proof?

In the justification of why cross-multiplication works, steps involving the associative property were omitted so that the essential reasons (definition of division, multiplicative inverses, and multiplicative identity) for the procedure would be more transparent.

Soccer referees are instructed by FIFA Law not to blow the whistle for every piddling foul, as it disrupts the flow of the game. By the same token, mathematicians do not include piddling details in proofs, as they disrupt the flow of the proof and conceal the main arguments.

MATH BACKGROUND

Interpreting the Minus Sign	
Here are three ways to interpret the minus sign, along with some examples.	
Operation Interpretation When the minus sign is between two expressions, it means “subtract the second expression from the first.”	Example: 5 – 3 The phrase “5 minus 3” can be read: • 5 take away 3, or subtract 3 from 5  Subtract or take away. There are 2 triangles remaining. • The difference between 5 and 3  Difference between two sets. There are 2 more circles than squares.
Geometric Interpretation In front of a number or variable, a minus sign means “opposite.” Geometrically, minus can be thought of as a reflection or mirror image. In this case, we are reflecting the number line through zero.	Example: -3 The phrase “minus 3” can be read: • Negative 3  Pictorially, this is a location on the number line that is 3 units left of zero. • Opposite of 3  This is the value you get by first locating 3 on the number line, and then locating that same distance on the opposite side of zero. Geometrically, minus can be thought of as a reflection or mirror image. In this case, the reflection of 3 through zero is -3.
Algebraic Interpretation The minus sign is used to show additive inverses. The identity $a + (-a) = 0$ means that $-a$ is the additive inverse of a. It is what we add to a to get 0.	Example: If $a = -3$, then $-a = 3$ The statement, “If a is equal to minus 3, then minus a is equal to 3” can be read: • If a is equal to the opposite of 3, then the opposite of a is equal to 3. When we add -3 and 3, the result is 0  Be careful not to read the variable $-a$ as “negative a” because the variable may not represent a number less than zero. However, reading $-a$ as “the opposite of a” is appropriate. For example, -(-3) = 3 may be read “the opposite of the opposite of 3 is 3.

MATH BACKGROUND

Why Does (-1)(-1) = 1?

We can use properties of arithmetic and equality to prove this fact.

Equation	Property
$(-1)(-1) + (-1)(1) = (-1)(-1 + 1)$	distributive property
$(-1)(-1) + (-1)(1) = (-1)(0)$	additive inverse property
$(-1)(-1) + (-1)(1) = 0$	multiplication property of zero
$(-1)(-1) + (-1) = 0$	multiplicative identity property
$(-1)(-1) + (-1) + 1 = 0 + 1$	addition property of equality (equals added to equals are equal)
$(-1)(-1) + 0 = 0 + 1$	additive inverse property
$(-1)(-1) = 1$	additive identity property (applied twice)

Derivation of Sign Rules for Multiplication

The multiplication table for +1 and -1 leads to the four sign rules for multiplication of integers:

- $(+1)(+1) = +1$ → (pos)(pos) = (pos),
- $(+1)(-1) = -1$ → (pos)(neg) = (neg),
- $(-1)(+1) = -1$ → (neg)(pos) = (neg),
- $(-1)(-1) = +1$ → (neg)(neg) = (pos).

The first of these rules follows from the definition of multiplication of positive integers.

Toward deriving the other rules, suppose that $a > 0$ and $b > 0$.

To establish the second rule, note that $(-a)(b) = (-1)(a)(b) = (-1)(ab) = -ab < 0$.

The third rule follows from the second rule and commutativity: $(a)(-b) = (-b)(a) = -ba < 0$.

Finally, the fourth rule follows from the identity $(-1)(-1) = +1$ and the equalities

$(-a)(-b) = (-1)(a)(-1)(b) = (-1)(-1)ab = 1ab = ab > 0$.

The sign rules extend to multiplication of rational numbers, and to multiplication of real numbers.

Division Defined in Terms of Multiplication

The operation of division is defined in terms of multiplication. This means that any statement about division can be reinterpreted as a statement about multiplication. Thus, to prove any assertion about division, it suffices to prove a corresponding assertion in terms of multiplication. In terms of a mathematical expression,

$$\frac{a}{c} = b \quad \text{exactly when} \quad a = b \bullet c \ \ (c \neq 0).$$

Indeed, $\frac{a}{c} = b$ means by definition that $a \bullet \frac{1}{c} = b$, where $\frac{1}{c}$ is the multiplicative inverse of c.

Since $a = a \bullet 1 = a \bullet \frac{1}{c} \bullet c$, this occurs exactly when $a = b \bullet c$.

Why is Division by Zero Undefined?

In mathematics, division by zero is undefined. This is a consequence of the laws of arithmetic, which guarantee that the number 0 has no multiplicative inverse. There are various ways of understanding why we should not be able to divide by zero.

A mathematical explanation:

> The definition of “division by *b*” is multiplication by the multiplicative inverse (reciprocal) of *b*. The number zero has no multiplicative inverse – there is no number a satisfying $0 \bullet a = 1$, because $0 \bullet a = 0$ for all a. Consequently, we cannot divide by zero.

A numerical example:

> $\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$ but what about $\begin{array}{r} \square \\ 0\overline{)12} \end{array}$?
>
> Since multiplication and division are inverses of one another, we know that the division fact $12 \div 4 = 3$ is related to the multiplication fact $4 \bullet 3 = 12$.
>
> Now consider $12 \div 0 = \square$. This leads to a multiplication fact $0 \bullet \square = 12$. What can be multiplied by 0 to obtain 12? The answer, of course, is that no such number exists.

A real-life example:

> Say there are 12 students that will be placed in groups of 4. Dividing $12 \div 4$ tells us that 3 groups of 4 can be formed.
>
> Now there are 12 students that will be placed in groups of 0. Dividing $12 \div 0$ tells us that $\square$ groups of 0 can be formed. How many groups of 0 make 12? Again, no such number exists.

An explanation using a pattern:

$$\frac{10}{1} = 10, \quad \frac{10}{0.1} = 100, \quad \frac{10}{0.01} = 1{,}000, \quad \frac{10}{0.001} = 10{,}000, \text{ etc.}$$

> As the denominator value gets smaller and smaller, the quotient value gets bigger and bigger. So, $\frac{10}{0}$ is infinite (in some sense). And so there is no number that satisfies the equation $\frac{10}{0} = \square$.

Order of Operations

There are many mathematical conventions that enable us to communicate better about common situations. For example, when using the coordinate plane, an agreed-upon convention is that we generally call the horizontal axis the *x*-axis, and the vertical axis the *y*-axis. This allows for easier, common communication.

There are agreed-upon rules as well for interpreting and simplifying arithmetic and algebraic expressions. We call these rules the standard order of operations. In a nutshell, the standard convention for order of operations is to simplify first the expressions within grouping symbols and exponents, then to perform multiplications and divisions, and finally additions and subtractions. More specifically:

1. Do the operations in grouping symbols first (e.g., use rules 2 – 4 inside parentheses, braces, brackets, and above and below fraction bars).
2. Calculate all the exponential expressions.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

Example:

$$\begin{aligned}\frac{3^2 + (6 \bullet 2 - 1)}{5} &= \frac{3^2 + (12 - 1)}{5}\\ &= \frac{3^2 + (11)}{5}\\ &= \frac{9 + (11)}{5} = \frac{20}{5} = 4\end{aligned}$$

There are many times for which these rules make complete sense and are quite natural. Take this case, for example:

You purchase 2 bottles of water for $1.50 each and 3 bags of peanuts for $1.25 each. Write an expression for this situation, and simplify the expression to find the total cost.

Expression: $\underbrace{2 \bullet 1.50}_{3.00} + \underbrace{3 \bullet 1.25}_{3.75}$

Simplification: $3.00 + 3.75 = \$6.75$

In this problem, it is natural to find the cost of the 2 bottles of water and then the cost of the 3 bags of peanuts prior to adding these amounts together. In other words, we perform the multiplication operations before the addition operation.

Note, however that if we were to perform the operations in order from left to right (as we read the English language from left to right) we would obtain a different result.

WRONG: $2 \bullet 1.50 + 3 \bullet 1.25 = ((2 \bullet 1.50) + 3) \bullet 1.25 = (3 + 3) \bullet 1.25 = 6 \bullet 1.25 = 7.50.$

MATH BACKGROUND

Why Does a Circle Have 360 Degrees?

Lack of sources makes it impossible to determine how various enumeration systems originated. There has been a great deal of speculation about why various systems were devised and how they interacted.

We know virtually nothing about the systems of enumeration used by the Sumerian civilization in Mesopotamia (before say 2200 BCE). However, we have extensive evidence of the systems of enumeration used by the subsequent old Babylonian civilization, going back past the Code of Hammurabi (1800 BCE). The evidence comes from excavations of mathematical cuneiform tablets with extensive records of warehouse inventories and also astronomical observations and calculations.

The two most important features of Babylonian mathematics are:

1. a place value system of recording numbers, and
2. use of a base-60 system of enumeration.

The use of a base-60 system is convenient because 60 has so many divisors. One theory about how this system evolved is based on currency conversions. In antiquity there were numerous currency systems, with different bases (think of pounds, shillings, and pence). It might have been natural for some merchants in trading centers to adopt a unit that would facilitate conversion rates among various currencies.

Whatever the origin of the base-60, the choice was serendipitous in the sense that it facilitated many mathematical calculations, such as calculations of reciprocals. In the golden era of Mesopotamian astronomy (final four centuries BCE), the Babylonian astronomers had a big advantage over the Egyptians, who were tied to a system of unit fractions that did not facilitate the intricate calculations required by astronomers.

The number 360 of degrees in a circle is a relatively recent development, stemming from the golden age of Mesopotamian astronomy. One explanation for the number 360 is that it might have been natural to approximate a circle (think in terms of the constellations through which the moon and planets pass) by chords that are sides of a regular hexagon composed of 6 equilateral triangles, thereby dividing the skies into 6 sectors. In a base-60 system, it would be natural to divide further the central angle of each triangle into 60 degrees, thereby leading to a total of 6 times 60, or 360, degrees in the circle. Further subdivision leads to 60 minutes in a degree, and 60 seconds in a minute.

A convenient property of the number 360 is that it is close to the number of days in a calendar year. The sun advances across the firmament by close to $\frac{1}{360}$ of a turn each day. The Egyptian calendar, claimed to have been adopted before 4000 BCE, divided the year into 12 months of 30 days for a total of 360 days, plus 5 extra days added to the end of the cycle. This scheme was modified by the Romans (Julian calendar of 12 months with a total of 365 days), and it was fine-tuned by Pope Gregory XIII (Gregorian calendar, with a 400-year cycle of prescribed leap years).

Different Definitions for Quadrilaterals

Is a rectangle a trapezoid? Is a square a rhombus? To a certain extent, the answers to these questions are a matter of convention and historical whim. According to the definitions we use, a parallelogram is a trapezoid, and a square is a rhombus. However, there is room for disagreement. Some textbook writers do not allow parallelograms to be trapezoids, and some textbook writers do not allow squares to be rhombi.

To complicate the issue further, definitions sometimes vary from one country to another. In England, a “trapezium” is a trapezoid, while in the United States a “trapezium” is a quadrilateral in which no two sides are parallel.

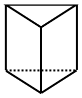
Polyhedra

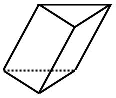
A polyhedron (plural: polyhedra) is a closed figure in space consisting of a finite number of polygonal figures that are joined at edges. Each polygonal figure is a face of the polyhedron, each side of a face is an edge of the polyhedron, and each vertex of a face is a vertex of the polyhedron.

The word “polyhedron” is used occasionally to refer to the solid figure comprised of the polyhedron and the region it encloses. Thus the volume of a polyhedron refers to the volume of the solid figure enclosed by the polyhedron.

A prism is a polyhedron that has two congruent and parallel polygons as bases, and lateral faces that are parallelograms. Prisms in which the bases are perpendicular to the lateral faces are called right prisms. The lateral faces of right prisms are rectangles.

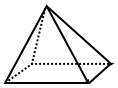
right triangular prism





oblique triangular prism

A pyramid is a polygon with a base that can be any polygon, one further vertex called the apex, and lateral faces that are triangles with a vertex at the apex and opposite sides the edges of the base.



Prisms and pyramids are named for the shape of their base(s). For example, a prism with a triangular base is called a triangular prism. A pyramid with a square base is called a square pyramid. A right prism with a rectangular base is called a right rectangular prism.

Many polyhedra (oriented in space) have a “bottom” and a “top,” and these are sometimes both referred to as the bases. The faces of a polyhedron other than its bases are referred to as its “sides.” The area of the sides of a polyhedral figure is referred to as its lateral area. A rectangular prism sitting on a table has six faces, including its top, its bottom, and its four sides. A pyramid sitting on a table has a base (the bottom) and several sides meeting at the apex. The lateral area of a pyramid is the area of the sides meeting at the apex.

There is an analogous nomenclature for cylinders. The bases of a cylinder are the two disks at either end, and the lateral area of the cylinder is the area of the curved tubular surface connecting these disks.

MATH BACKGROUND

The Number Pi

Pi (usually written as the Greek letter π) is the value of the ratio of the circumference of a circle to its diameter. The constant π is slightly greater than 3, so that the circumference of a circle is a little more than 3 times its diameter.

Various approximations to the value of the ratio of the circumference to the diameter of a circle have been used by many civilizations over the centuries. The oldest written approximations to π , dating from 1600 – 1900 BCE and accurate to within 1%, come from Babylonic cuneiform tablets and an Egyptian papyrus (the Rhind papyrus).

The Greek mathematician Archimedes (third century BCE) found some very accurate approximations to π . In his manuscript *Measurement of a Circle*, Archimedes showed that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. We often use the approximation $3\frac{1}{7} = \frac{22}{7}$ for the value of π today. Other common approximations to π are 3.14 and 3.1416.

The notation π for pi was first used around 1700. It probably was meant to stand for "periphery." The notation was adopted by Euler in the mid-1700s, and subsequently came into common usage. That same century, the Swiss mathematician, J.H. Lambert, established that π is not a rational number, that is, π is not a quotient of integers. Finally, in 1882, a German mathematician, Ferdinand Lindemann, succeeded in proving that π is a transcendental number, that is, π is not the solution of any algebraic equation with integer coefficients. By doing this, he settled the last of the three classical Greek problems, showing that it is impossible to "square a circle" (construct a square of the same area as a given circle) using a ruler and compass.

When Is a Proof a Proof?

The investigation where students establish for the formula for the area of a circle is certainly convincing. Is it a proof? Yes and no. It does not constitute a formal proof, yet it contains the essential line of reasoning of a correct proof. A person with a sophisticated background in mathematical analysis (epsilons and deltas) may look at the argument and say that yes, it is a proof, in the sense that he or she can translate to the language of the analyst to form a correct formal proof.

Even mathematicians rarely produce complete formal proofs of their theorems. Hyman Bass suggests:

> "Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists."

This statement sheds light on how we might treat mathematical reasoning and proof in the early grades—what we might expect, and what we should not expect.

In any event, the idea of proof is a flexible notion. The conventions and standards of the mathematical community—whether that community is the group of students in a classroom or the readers of a mathematics research journal—play a role in deciding whether an argument is a valid proof.

"Mathematics and Teaching," in *I, Mathematician*, edited by P. Cassazza, S,G.Krantz, and R.D. Ruden, MAA, 2015

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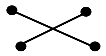
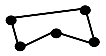
Polygons

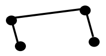
A polygon is a simple closed curve in a plane formed by laying straight line segments end-to-end. The line segments are the sides (or edges) of the polygon, and their endpoints are the vertices of the polygon. By “closed,” we mean that the final endpoint of the final segment laid down is the initial endpoint of the first segment laid down. By “simple,” we mean that each edge meets only two other edges, namely, the preceding edge and the succeeding edge at the vertices where they are joined.



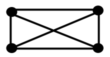
not closed
not a polygon



polygon



not closed
not a polygon



not simple
not a polygon

A polygon divides the plane into a bounded region (the inside of the polygon) and an unbounded region (the outside of the polygon). The inside of the polygon is referred to as a polygonal figure. We use the word “polygon” also to refer to this polygonal figure, and the meaning of “polygon” can sometimes only be inferred from context. The area of a polygon refers to the area of the polygonal figure enclosed by the polygon.

Base, Altitude, and Height

When a polygon is represented visually, it is often pictured as sitting on one of its sides. It is natural to call that side the "base" of the polygon. If we rotate the polygon, it may be sitting on another side, and we should refer to that other side as its base. Mathematically, the specification of the base of a polygon is arbitrary. We specify beforehand which side is the base, irrespective of how the polygon might be represented visually, and we develop definitions and formulas for a polygon with a predesignated base.

In the case of a rectangle, we may designate any side as the base of the rectangle. In the formula

Area = (length) × (width),

the "length" then refers to the length of the base, while the "width" refers to the length of the sides perpendicular to the base.

An altitude of a parallelogram on a given base is a line segment perpendicular to the base and connecting a point on the base (extended if necessary) to a point on the opposite side. The height of the parallelogram is the length of the altitude. The area of a parallelogram is given by

Area = (length of base) × (height).

Though the factors in this product depend on which side of the parallelogram is selected to be the base, the area will be the same no matter which side we designate as the base. In the case of a triangle, there are three choices for base, three different altitudes, and possibly three different heights. The formulas for area all give the same result.

Something similar happens for polyhedra. We may designate any face of a rectangular prism as the base of the prism. The "height" then refers to the length of the edges perpendicular to the base, and the formula

Volume of prism = (area of base) × (height)

is valid, no matter which face we designate as the base.

Notation Clarification — "*B*" vs. "*b*"

In mathematics, the names we assign to variables can be quite arbitrary, and wide ranges of letters and symbols are available to denote variables. While names can be arbitrary, in practice we aim for clarity by selecting names for variables that remind us what they are and that are consistent with standard usage. For example, we typically refer to the axes of the coordinate plane as the x-axis and y-axis. In algebra, we traditionally denote the unknown by x, while we use a, b, and c to denote constants. The slope of a line is often denoted by m. The volume of a solid is often denoted by V.

In geometry, lengths are usually denoted by lower case letters, while areas and volumes are denoted by upper case letters. Thus, the formula for the area of a parallelogram is often given as $A = bh$, where b is the length of the base and h is the height (length of an altitude).

When establishing the volume V of a right rectangular prism, we use an upper case B since it represents an area, namely the area of the base of the prism. This distinguishes it clearly from the b used in the area formula $A = bh$, in which b represents a length.

$V = Bh$, where B = area of base and h = height of the prism

Most important is that variables are defined properly and used consistently.

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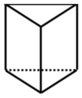

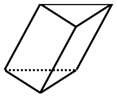

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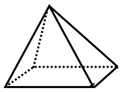

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MATH BACKGROUND

Using Data to Understand Our World

A key to understanding the world we live in is to collect data: temperature, wind velocity, rainfall, water levels, animal populations, cases of malaria, and so on. The data have a story to tell. It is the job of mathematics and statistics to read the story. For example, counting animals with a method such as mark-recapture can inform decisions about habitat protection, breeding programs, and other plans to protect endangered species in the environment. We need mathematics and statistics to tell us what the trends are and with what certainty. Once we've read the story, it is our job to take steps to make the world a better place to live in.

"The Quartile" or "In the Quartile"

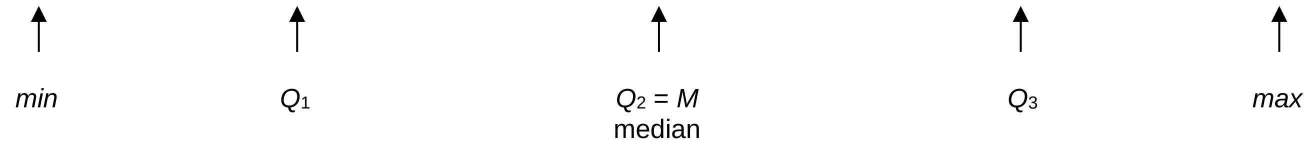
The word "quartile" is used in statistics in two different ways. Most often, it is used to denote numbers that separate the data set into four equal parts. In the sample data set below, $Q_1 = 2$, $Q_2 = 3$, and $Q_3 = 4.5$.

The word "quartile" can also refer to the set of values in one of the four equal parts of the data set. In the sample data set below the "fourth quartile" (or "top quartile") is {5, 5, 7}, and the "first quartile" (or "bottom quartile") is {1, 1, 2}.

Thus, "the first quartile" is 2, but "the value 2 lies both in the "first quartile" and in the "second quartile."
This ambiguous use of terms occurs often in mathematics, and the precise meaning must be determined from context. For instance, a "triangle" is defined to be three segments joined end to end to enclose an area, but the word "triangle" may also refer to the triangular area enclosed by the three segments.

Sample data set:

1	1	2	2	2	2	3	3	3	4	5	5	7



Mean Absolute Deviation (MAD)

The standard deviation σ is the measure of spread of a numerical data set preferred by statisticians. It occurs naturally in many applications, such as in the analysis of variants. However, the standard deviation is difficult for middle school students to calculate by hand, since it has some squares and a square root.

The mean absolute deviation (MAD) is an alternative measure of spread, which has the same general form as the standard deviation but is easier to calculate by hand. The MAD is computed by adding the distances between each data value and the mean, and then dividing by the number of data values. It turns out that $\sigma \geq \text{MAD}$. In fact, $\sigma > \text{MAD}$ except in the very special cases that there is only one data value, or that there are exactly two data values, each with probability one half. There are data sets for which σ is quite large while MAD is close to zero.

For a normal distribution, σ is obtained by multiplying the MAD by the square root of 2π, $\sigma = \sqrt{\frac{\pi}{2}} \bullet \text{MAD}$.

Sampling

Sampling refers to selecting a subset of a population to be examined for the purpose of drawing statistical inferences about the entire population. If the sample is representative of the entire population, we may make valid inferences about the entire population based on properties of the sample.

Suppose we want to know how many hours per week students in a school district spend watching television. From the population of all students, we select a sample and we ask the students in the sample how many hours they watch television. We would like to infer that the average time spent watching TV for all students is about the same as for students in the sample.

Sampling has two goals that are often in competition and must be balanced.

1. The selection of the sample should fit into a mathematical framework that provides information on the general population. The typical sort of information we seek is a probabilistic statement that with 95% certainty the average number of hours that students in the school district watch TV each week is within (say) an hour of the sample average.
2. The method of selecting the sample should be as inexpensive as possible, in terms of both money and time. For instance, we might like to select a sample that has the least possible number of students.

The easiest way to select a sample might be to enter the first classroom we happen upon and ask students in that class how many hours they watch TV. Such a sample is called a convenience sample. It takes a minimal amount of time, and it is inexpensive. However, the students in the classroom may not be representative of all students, and the results of the sample cannot be used to draw inferences about the general population. At best, the results can provide some clues to the TV-watching habits of students.

To select a more representative sample, we might place the names of all students in the district in a hat, mix the names thoroughly, and have a blindfolded person draw names from the hat for the sample. After each name is drawn, we replace it in the hat, and we remix the names before the next draw. Such a random sample is called a simple random sample. Its mathematical properties allow us to draw inferences along probabilistic lines that are valid for all students.

A variation of the above procedure would be to discard a name once it is drawn, so that we do not get data from the same student twice. This sampling procedure of drawing without replacement does not qualify as simple random sampling. However, if the number of students in the sample (sample size) is very small compared to the number of students in the entire population, the probability of the same name being drawn more than once is very small, and the statistics obtained on the basis of sampling without replacement will not differ significantly from those obtained by sampling with replacement.

Sampling can be immensely complicated. There is a multitude of sampling techniques and variations. Often, sampling techniques must be adapted for the problem at hand. Two typical examples are stratified sampling and clumped sampling.

Stratified sampling: If the population can be divided into subpopulations, or strata, that are relatively homogeneous, then statistical accuracy can be improved by sampling each stratum independently. For instance, to obtain information on TV watching, it might be advantageous to break the student population into high school students, middle school students, and elementary school students, and to select a simple random sample from each of these strata.

Clumped sampling: Instead of taking a random sample of the entire population, the population is divided into clumps, a sample of clumps is selected at random, and each clump in the sample is inspected. For instance, to get information Statewide about TV watching, it might be expensive and time-consuming to collect the names of all students in the State and to select a random sample from this large population. Instead, we might select a random sample of schools in the State and sample only those schools.