TEACHER EDITION

MathLinks GRADE 7

COMPILED STUDENT RESOURCES

STUDENT RESOURCES

Word or Phrase	Definition
dependent events	Two events are dependent if the occurrence (or nonoccurrence) of one event affects the likelihood of the other. See independent events
event	An event is a subset of the sample space. See sample space. In the probability experiment of rolling a number cube, "rolling an even number" is an event, because getting a 2, 4, or 6 is a subset (part) of the sample space of {1, 2, 3, 4, 5, 6}.
experimental probability	In a repeated probability experiment, the experimental probability of an event is the number of times the event occurs divided by the number of trials. This is also called empirical probability. If, in 25 rolls of a number cube, we obtain an even number 11 times, we say that the experimental probability of rolling an even number is $\frac{11}{25} = 0.44 = 44\%$.
fair game	A game of chance is a fair game if all players have equal probabilities of winning. A two-person game of chance is a fair game if each player has probability $\frac{1}{2}$ of winning, that is, if each player has the same probability of winning as of losing.
independent events	Two events are independent if the occurrence (or nonoccurrence) of one event does not affect the likelihood of the other. See dependent. In the probability experiment of rolling a number cube and flipping a coin, the event of rolling a 1 is independent of the event of getting heads on the coin flip. The probability of rolling the 1 is $\frac{1}{6}$, no matter what the outcome of the coin flip is. In other words, the cube roll does not depend at all on the coin flip.
outcome	An outcome is a result of a probability experiment. If we roll a number cube, there are six possible outcomes: 1, 2, 3, 4, 5, 6.
percent	A percent is a number expressed in terms of the unit $1\% = \frac{1}{100}$. Fifteen percent = $15\% = \frac{15}{100} = 0.15$. $\frac{5}{6} = 0.8\overline{3} = 83.\overline{3}\%$

Word or Phrase	Definition
probability	The probability of an event is a measure of the likelihood of that event occurring. The probability $P(E)$ of the event E occurring satisfies $0 \le P(E) \le 1$. If the event, E, is certain to occur, then $P(E) = 1$. If the event E is impossible, then $P(E) = 0$. When flipping a fair coin, the probability that it will land on heads is $\frac{1}{2} = 0.5 = 50\%$.
probability experiment	A probability experiment is an experiment in which the results are subject to chance. Rolling a number cube can be considered a probability experiment.
repeating decimal	A repeating decimal is a decimal that ends in repetitions of the same block of digits. The repeating decimal 52.19343434… ends in repetitions of the block "34." An abbreviated notation for the decimal is $52.19\overline{34}$, where the bar over 34 indicates that the block is repeated. The terminating decimal 4.62 is regarded as a repeating decimal. Its value is 4.620000…
sample space	The sample space for a probability experiment is the set of all possible outcomes of the experiment. In the probability experiment of rolling a number cube, the sample space can be represented as the set {1, 2, 3, 4, 5, 6}.
simulation	Simulation is the imitation of one process by means of another process. We may simulate rolling a number cube by drawing a card blind from a group of six identical cards labeled one through six. We may simulate the weather by means of computer models.
terminating decimal	A terminating decimal is a decimal whose digits are 0 from some point on. Terminating decimals are regarded as repeating decimals, though the final 0's in the expression for a terminating decimal are usually omitted. See repeating decimal. 4.62 = 4.62000000… is a terminating decimal with value $4 + \frac{6}{10} + \frac{2}{100}$.
theoretical probability	The theoretical probability of an event is a measure of the likelihood of the event occurring. In the probability experiment of rolling a (fair) number cube, there are six equally likely outcomes, each with probability $\frac{1}{6}$. Since the event of rolling an even number corresponds to 3 of the outcomes, the theoretical probability of rolling an even number is 3 out of 6, or $3 \bullet \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.
trial	Each performance or repetition of a probability experiment is called a trial. Flipping a coin 25 times can be viewed as 25 trials of the probability experiment of flipping a coin once.

Phrases That Describe Probabilities

In their assessment reports on climate change, climate scientists attach the following probabilities to common expressions of likelihood:

Virtually certain:	> 99% probability
Extremely likely:	> 95% probability
Very likely:	> 90% probability
Likely:	> 66% probability
More likely than not:	> 50% probability
About as likely as not:	33 to 66% probability
Unlikely:	< 33% probability
Very unlikely:	< 10% probability
Extremely unlikely:	< 5% probability
Exceptionally unlikely:	< 1% probability

Estimating Probabilities from an Experiment With Equally Likely Outcomes

To estimate the probability of an event E, repeat the experiment a number of times and observe how many times the event occurs. The estimate for the probability of the event E occurring is then given by the fraction:

$$\text{estimate} = \frac{\text{number of times an event } E \text{ occurs}}{\text{number of trials}} = \frac{\text{numerator}}{\text{denominator}}$$

In a probability experiment of rolling a number cube with six equally likely outcomes, each has probability $\frac{1}{6}$.

The event of rolling an odd number corresponds to three outcomes: 1, 3, or 5. Below is data from an experiment where a cube is rolled 10 times.

Trial #	1	2	3	4	5	6	7	8	9	10
Outcome	4	5	6	3	5	2	1	6	4	2

In this experiment, an odd number occurred 4 times.

$$\text{estimate(odd)} = \frac{4}{10} = \frac{2}{5} = 40\%$$

Since the estimate is based on an experiment, different experiments may lead to different estimates.

Finding Theoretical Probabilities

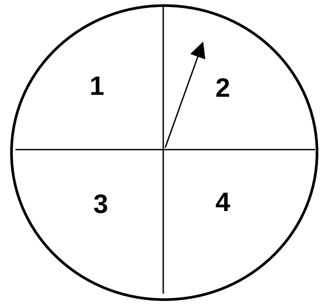
In a probability experiment of rolling a number cube with six equally likely outcomes, each has probability $\frac{1}{6}$.

The event of rolling an odd number corresponds to three outcomes: 1, 3, or 5. Thus the theoretical probability of rolling an odd number is given by the fraction:

$$P(E) = \frac{\text{number of outcomes in an event } E}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2} = 50\%$$

Sample Space Displays

Suppose our experiment is to flip a coin and then spin the spinner.

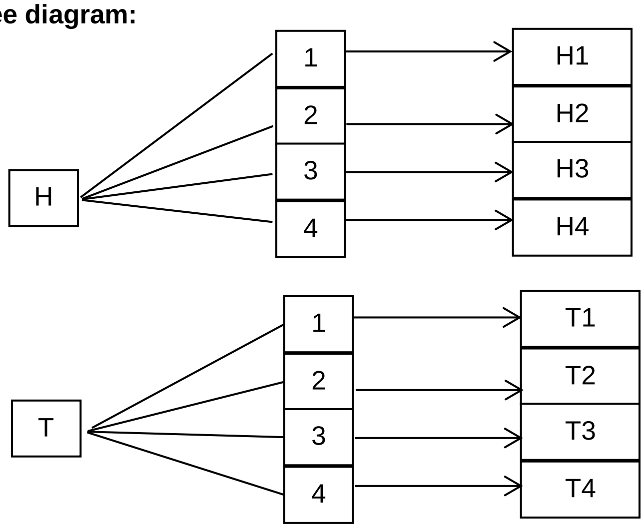


Below are three ways to show all the outcomes (or the sample space) of the experiment.

1. Outcome grid:

		Spinner			
		1	**2**	**3**	**4**
Coin Flip	**Heads (H)**	H1	H2	H3	H4
	Tails (T)	T1	T2	T3	T4

2. Tree diagram:



3. List

H1 H2 H3 H4
T1 T2 T3 T4

STUDENT RESOURCES

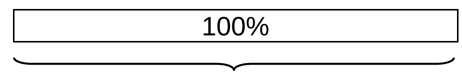
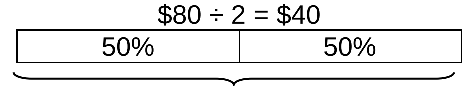
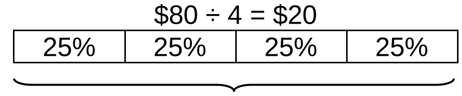
Word or Phrase	Definition
decrease in a quantity	The decrease in a quantity is the original value minus the new value. The percent decrease in a quantity is the value of the ratio of the decrease to the original quantity, expressed as a percent. Last year, there were 200 students in the school. This year, there are 178 students in the school. The decrease in the number of students is 200 – 178 = 22. Since $\frac{22}{200} = \frac{11}{100}$, the percent decrease is 11%.
discount	The discount (or markdown) of an item is the decrease in the price of the item; that is, the original price of the item minus the new price. The percent discount is the percent decrease in the price of the item; that is, the value of the ratio of the decrease to the original value, expressed as a percent. Last week, the price of an MP3 player was $200. This week, the price is $178. The discount is 200 – 178 = 22. Since $\frac{22}{200} = \frac{11}{100}$, the percent discount is 11%.
increase in a quantity	The increase in a quantity is the new value minus the original value. The percent increase in a quantity is the value of the ratio of the increase to the original quantity, expressed as a percent. Last year there were 200 students in school. This year, there are 208 students. The increase in the number of students is 208 – 200 = 8. Since $\frac{8}{200} = \frac{4}{100}$, the percent increase is 4%.
markup	The markup on an item is the increase in the price of the item, that is, the new price of the item minus the original price. The percent markup is the percent increase in the price of the item. Last week, the price of an MP3 player was $200. This week, the price is $208. The markup is 208 – 200 = 8. Since $\frac{8}{200} = \frac{4}{100}$, the percent markup is 4%.
percent	A percent is a number expressed in terms of the unit 1% = $\frac{1}{100}$. To convert a positive number to a percent, multiply the number by 100. To convert a percent to a number, divide the percent by 100. 4 = 4 $\times$ 100% = 400%. Fifteen percent = 15% = $\frac{15}{100}$ = 0.15.

Word or Phrase	Definition
percent decrease in a quantity	See decrease in a quantity.
percent increase in a quantity	See increase in a quantity.
percent of a number	A percent of a number is the product of the percent and the number. It represents the number of parts per 100 parts. 15% of 300 is $\frac{15}{100} \bullet 300 = 45$. If 45 out of 300 students are boys, then 15 out of every 100 students are boys, and 15% of the students are boys.
ratio	A ratio is a pair of positive numbers in a specific order. The ratio of a to b is denoted by $a : b$ (read "*a* to *b*," or "*a* for every *b*"). The ratio of 3 to 2 is denoted by 3 : 2. The ratio of dogs to cats is 3 to 2. There are 3 cups of water for every 2 cups of juice. The fraction $\frac{3}{2}$ does not represent this ratio, but it does represent the *value of the ratio* (or the unit rate).
scale	In a scale drawing of a figure, the scale is the ratio of lengths in the scale drawing to lengths in the actual figure. The blueprint of a house floorplan has a scale of 1 inch to 5 feet, or 1 in : 5 ft. Each inch on the blueprint represents 5 feet. The map has a scale of 3 centimeters to 10 kilometers, or 3 cm : 10 km. Each 3 centimeters on the map represents 10 kilometers.
scale drawing	A scale drawing of a geometric figure is a drawing in which all lengths have been multiplied by the same scale factor. A blueprint (drawing to scale) of a house floorplan is a scale drawing.
scale factor	A scale factor is a positive number which multiplies some quantity. To make a scale drawing of a figure, we multiply all lengths by the same scale factor. If the scale factor is greater than 1, the drawing is an enlargement, and if the scale factor is between 0 and 1, the drawing is a reduction.

Some Fraction-Decimal-Percent Equivalents		
$\frac{1}{2} = \frac{50}{100} = 0.5 = 50\%$ $\frac{1}{4} = \frac{25}{100} = 0.25 = 25\%$ $\frac{3}{4} = \frac{75}{100} = 0.75 = 75\%$ $\frac{5}{4} = \frac{125}{100} = 125 = 125\%$ Conversion strategy: Think: $\frac{3}{4}\left(\frac{25}{25}\right) = \frac{75}{100} = 75\%$	$\frac{1}{10} = \frac{10}{100} = 0.1 = 10\%$ $\frac{3}{10} = \frac{30}{100} = 0.3 = 30\%$ $\frac{5}{10} = \frac{50}{100} = 0.5 = 50\%$ Conversion strategy: Think: $\frac{3}{10} = \frac{30}{100}$, so 0.3 = 0.30 = 30%	$\frac{1}{25} = \frac{4}{100} = 0.04 = 4\%$ $\frac{16}{25} = \frac{64}{100} = 0.64 = 64\%$ $\frac{9}{50} = \frac{18}{100} = 0.18 = 18\%$ Conversion strategy: Think: 25(4) = 100, so $\frac{16}{25}\left(\frac{4}{4}\right) = \frac{64}{100} = 64\%$
$\frac{3}{20} = \frac{15}{100} = 0.15 = 15\%$ $\frac{13}{20} = \frac{65}{100} = 0.65 = 65\%$ $\frac{19}{20} = \frac{95}{100} = 0.95 = 95\%$ Conversion strategy: Think: 20 nickels in a dollar $\frac{1}{20}$ of a dollar is \$0.05	$\frac{1}{5} = \frac{2}{10} = 0.2 = 20\%$ $\frac{2}{5} = \frac{4}{10} = 0.4 = 40\%$ $\frac{3}{5} = \frac{6}{10} = 0.6 = 60\%$ $\frac{4}{5} = \frac{8}{10} = 0.8 = 80\%$ Conversion strategy: Think: If I know tenths, I can easily convert to hundredths.	$\frac{1}{8} = \frac{12.5}{100} = 0.125 = 12.5\%$ $\frac{3}{8} = \frac{37.5}{100} = 0.375 = 37.5\%$ $\frac{5}{8} = \frac{62.5}{100} = 0.625 = 62.5\%$ $\frac{7}{8} = \frac{87.5}{100} = 0.875 = 87.5\%$ Conversion strategy: Think: $\frac{1}{4} = \frac{25}{100}$, so half of $\frac{1}{4}$ is $\frac{1}{8} = \frac{12.5}{100}$ = 12.5%

Using "Chunking Strategies" to Find Percents of Numbers

We use the word "chunking" to describe a process of decomposing and composing numbers to make calculations easier, especially when done mentally. Another way to describe this is "taking numbers apart and putting them back together." For example, if adding 17 and 26, we might decompose each number into tens and ones, adding 10 + 20 = 30, and 7 + 6 = 13, and finalizing the sum by adding 30 + 13 = 43.

Think	Example
Finding 100% of something is the same as finding all of it.	100% of \$80 = \$80 
Finding 50% of something is the same as finding half of it. This is the same as multiplying by $\frac{1}{2}$ or dividing by 2.	50% of \$80 = $\frac{1}{2}$(\$80) = \$40 \$80 ÷ 2 = \$40 
Finding 25% of something is the same as finding one-fourth of it. This is the same as multiplying by $\frac{1}{4}$ or dividing by 4.	25% of \$80 = $\frac{1}{4}$(\$80) = \$20 \$80 ÷ 4 = \$20 
Finding 10% of something is the same as finding one-tenth of it. This is the same as multiplying by $\frac{1}{10}$ or dividing by 10.	10% of \$80 = $\frac{1}{10}$(\$80) = \$8 \$80 ÷ 10 = \$8
Finding 1% of something is the same as finding one-hundredth of it. This is the same as multiplying by $\frac{1}{100}$ or dividing by 100.	1% of \$80 = $\frac{1}{100}$(\$80) = \$0.80 \$80 ÷ 100 = \$0.80
Finding 20% of something is the same as doubling 10% of it.	20% of \$80 = 2(\$8) = \$16
Finding 5% of something is the same as halving 10% of it.	5% of \$80 = $\frac{1}{2}$(\$8) = \$4
Finding 15% of something is the same as adding 10% of it and 5% of it.	15% of \$80 = \$8 + \$4 = \$12

Using Multiplication to Find Percents of Numbers

Some percents are hard to find mentally. For example, finding 17% of something is the same as finding $\frac{17}{100}$ = 0.17 of it. In this case, it may be easier to find the percent by using the definition of a percent of a number: A percent of a number is the product of the percent and the number.

Find 17% of $80.

Strategy 1: **Use fractions**

$$\frac{17}{100} \bullet 80 = \frac{17 \bullet 80}{100} = \frac{1360}{100} = 13.60$$

So 17% of $80 is $13.60.

Strategy 2: **Use decimals**

$(0.17) \bullet (80) = 13.6$ or 13.60

So 17% of $80 is $13.60.

Percent Increase

Percent increases occur frequently as tips, taxes, and price markups. To find a percent increase, find the amount of the increase and add it to the original quantity.

Example	Original amount	Percent increase	Amount of increase	New amount (original + increase)
Leave a **tip** on a restaurant bill.	$40	20%	20% of $40 = $8	$40 + $8 = $48
Pay **tax** on a clothes purchase.	$50	8%	8% of $50 = $4	$50 + $4 = $54
Pay a **markup** on a video game.	$75	10%	10% of $75 = $7.50	$75 + $7.50 = $82.50

Percent Decrease

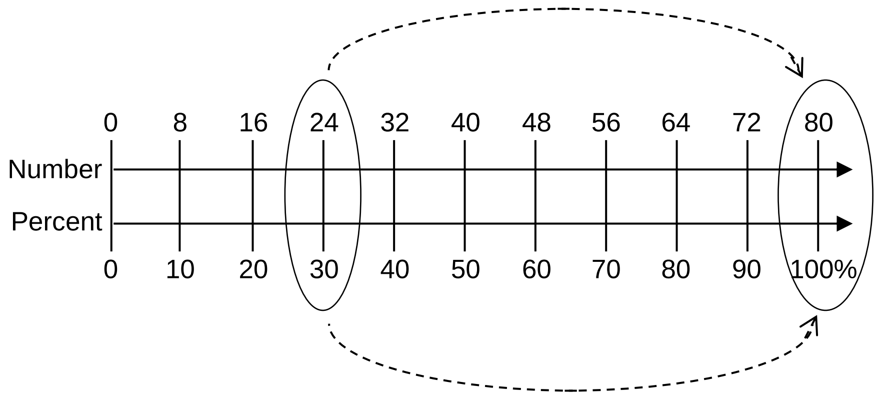
Percent decreases occur frequently as sales and discounts. To find a percent decrease, find the amount of the decrease and subtract it from the original quantity.

Example	Original amount	Percent decrease	Amount of decrease	New amount (original – decrease)
Sale on shoes purchase	$50	25%	25% of $50 = $12.50	$50 – $12.50 = $37.50
Discount on a dress	$90	40%	40% of 90 = $36.00	$90 – $36 = $54

Using Double Number Lines to Solve a Percent Problem:
30% of 80 is what amount?

Strategy 1: Solve on the double number line.

Create a double number line with percents represented in increments of 10% on the bottom line, and the whole number represented in increments on the top. Since the whole is 80 (in this case), count by 8s for the increments (80 ÷ 10 = 8).



Since 30% corresponds to 24 on the double number line, 30% of 80 is 24.

Strategy 2: Identify equivalent ratios on the double number line.

Create equations based on the part to whole ratio relationships.

$$\frac{\text{part}_{number}}{\text{whole}_{number}} = \frac{\text{part}_{percent}}{\text{whole}_{percent}}$$

$$\frac{\boxed{24}}{80} = \frac{30}{100}$$

This equivalence is based on the dotted arrows above.

Create equations based on the part to part ratio relationships.

$$\frac{\text{part}_{number}}{\text{part}_{percent}} = \frac{\text{whole}_{number}}{\text{whole}_{percent}}$$

$$\frac{\boxed{24}}{30} = \frac{80}{100}$$

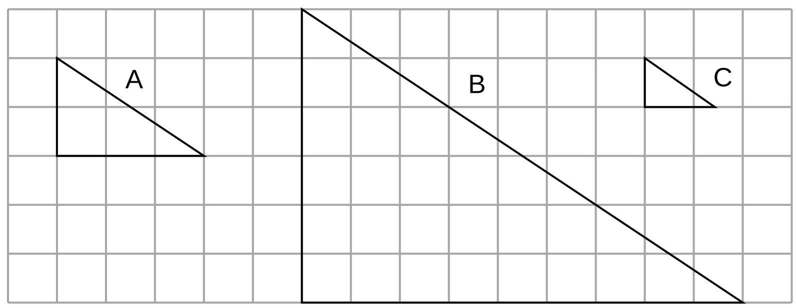
This equivalence is based on the circles above.

Scale Factors

Consider triangle A as the original figure.

To make Triangle B below, multiply each dimension of Triangle A by a scale factor of 3. Triangle B is a 300% enlargement of Triangle A. An enlargement is created when multiplying by a scale factor greater than 1.

To make Triangle C below, multiply each dimension of Triangle A by a scale factor of $\frac{1}{2}$. Triangle C is a 50% reduction of Triangle A. A reduction is created when multiplying by a scale factor between 0 and 1.



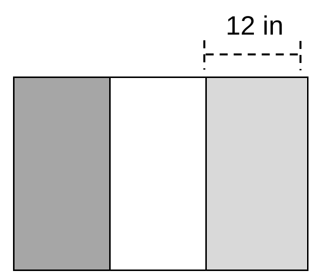
Scale Drawings

The flag of Italy is composed of three stripes (green, white, and red) that divide the flag into thirds. Pictured below is a scale drawing of the flag.

Suppose the original flag is 3 feet by 2 feet, and the scale drawing is 1.5 inches by 1 inch.

This scale may be represented as a ratio:

1.5 in	:	3 ft	→	1.5 in	:	36 in
1 in	:	2 ft	→	1 in	:	24 in
				1	:	24



The scale drawing is a reduction of the flag. The scale factor (value of the ratio) that produces this reduction is $\frac{1}{24}$. In other words, to obtain lengths for the drawing, multiplying the corresponding actual lengths by $\frac{1}{24}$.

STUDENT RESOURCES

Word or Phrase	Definition
complex fraction	A complex fraction is a fraction whose numerator or denominator is a fraction. Two complex fractions are $\frac{\frac{4}{5}}{\frac{1}{2}}$ and $\frac{\frac{1}{5}}{3}$.
constant of proportionality	See proportional.
dependent variable	A dependent variable is a variable whose value is determined by the values of the independent variables. See independent variable.
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8 + x}{10}$, and $4v – w$.
equivalent ratios	Two ratios are equivalent ratios if each number in one ratio is obtained by multiplying the corresponding numbers in the other ratio by the same positive number. 5 : 3 and 20 : 12 are equivalent ratios because both numbers in the ratio 5 : 3 are multiplied by 4 to get to the ratio 20 : 12.
independent variable	An independent variable is a variable whose value may be specified. Once specified, the values of the independent variables determine the values of the dependent variables. For the equation $y = 3x$, y is the dependent variable and x is the independent variable. We may assign a value to x. The value assigned to x determines the value of y.

<table>
<tr><th>Word or Phrase</th><th>Definition</th></tr>
<tr><td>input-output rule</td><td>An input-output rule for a sequence of values is a rule that establishes explicitly an output value for each given input value.

<table>
<tr><td>input value (x)</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>x</td></tr>
<tr><td>output value (y)</td><td>1.5</td><td>3</td><td>4.5</td><td>6</td><td>7.5</td><td>$1.5x$</td></tr>
</table>

In the table above, the input-output rule could be $y = 1.5x$. In other words, to get the output value, multiply the input value by 1.5. If $x = 100$, then $y = 1.5(100) = 150$.</td></tr>
<tr><td>proportional</td><td>Two variables are proportional if the values of one are the same constant multiple of the corresponding values of the other. The variables are said to be in a proportional relationship, and the constant is referred to as the constant of proportionality.

If Wrigley eats 3 cups of kibble each day, then the number of cups of kibble is proportional to the number of days. If x is the number of days, and y is the number of cups of kibble, then $y = 3x$. The constant of proportionality is 3.</td></tr>
<tr><td>proportional relationship</td><td>See proportional.</td></tr>
<tr><td>ratio</td><td>A ratio is a pair of positive numbers in a specific order. The ratio of a to b is denoted by $a : b$ (read "a to b," or "a for every b").

The ratio of 3 to 2 is denoted by $3 : 2$. The ratio of dogs to cats is 3 to 2. There are 3 cups of water for every 2 cups of juice. The fraction $\frac{3}{2}$ does not represent this ratio, but it does represent the ratio's value (or the unit rate).</td></tr>
<tr><td>unit price</td><td>A unit price is a price for one unit of measure.</td></tr>
<tr><td>unit rate</td><td>The unit rate associated with a ratio $a : b$ of two quantities a and b, $b \neq 0$, is the value $\frac{a}{b}$, to which units may be attached.

The ratio of 40 miles each 5 hours has unit rate of 8 miles per hour.</td></tr>
<tr><td>value of a ratio</td><td>See unit rate.</td></tr>
<tr><td>variable</td><td>A variable is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to quantities that vary in a relationship (as in a formula or an input-output rule). They may refer to unknown quantities in expressions, equations or inequalities. Finally, they may be used to generalize rules of arithmetic.

In the equation $d = rt$, the quantities d, r, and t are variables.
In the equation $2x = 10$, the variable x may be referred to as the unknown.
The equation $a + b = b + a$ generalizes the commutative property of addition for all numbers a and b.</td></tr>
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Testing for a Proportional Relationship

Here are three ways to test if two variables are in a proportional relationship:

- The values of the ratios (unit rates or unit prices) created by data pairs are equivalent.
- An equation in the form $y = kx$ fits all corresponding data pairs.
- Graphed data pairs fall on a line through the origin (0, 0).

Note that this example does **not** represent a proportional relationship. Alexa buys tickets when she goes to the amusement park. This chart shows the costs for different quantities of tickets.

# of tickets	10	20	25	50	100
total cost	$40	$60	$75	$125	$200
cost per ticket	$4	$3	$3	$2.50	$2

Since the costs per ticket (unit prices) are not the same, ticket purchasing at this amusement park does **not** represent a proportional relationship.

This example **does** represent a proportional relationship. Antonio kept track of the number of miles he traveled each time he filled his tank with gas. Here is some data.

number of miles	100	200	175	300
number of gallons	4	8	7	12
miles per gallon	25	25	25	25

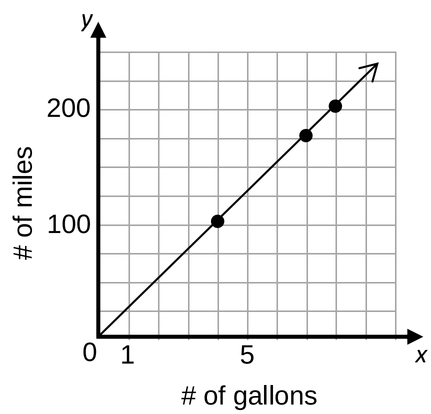
Since the miles per gallon (unit rates) created by the data pairs is the same, this situation represents quantities in a proportional relationship.

Furthermore,

Let x = the number of gallons
Let y = the number of miles

The data fits the equation $y = 25x$ (an equation in the form $y = kx$), which is an equation that represents a proportional relationship.

Finally, if the points for (gallons, miles) are graphed, they will fall on a line through the origin (0,0).



Multiple Representations and Proportional Relationships

Suppose 4 balloons cost $6.00 and each balloon is the same price. Here are some strategies for representing this proportional relationship.

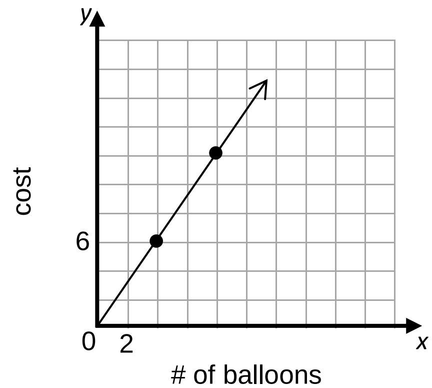
Strategy 1: Tables

Create a table to calculate unit rates. If the unit rates are the same, the variables are in a proportional relationship.

Number of Balloons	Cost	Unit Price
4	$6.00	$1.50
2	$3.00	$1.50
1	$1.50	$1.50
8	$12.00	$1.50

Strategy 2: Graphs

A straight line through the origin indicates quantities in a proportional relationship.



Strategy 3: Equations

An equation of the form $y = kx$ indicates quantities in a proportional relationship. In this case,

y = cost in dollars
x = number of balloons
k = cost per balloon (unit price)

To determine the unit price, create a ratio whose value is: $\frac{6 \text{ dollars}}{4 \text{ balloons}} = 1.50 \frac{\text{dollars}}{\text{balloons}}$

Therefore, k = $1.50 per balloon, and $y = 1.50x$.

This equation expresses the output as a constant multiple of the input, showing that the relationship is proportional.

Sense-Making Strategies to Solve Proportional Reasoning Problems

How much will 5 pencils cost if 8 pencils cost \$4.40?

Strategy 1: Use a "halving" strategy

If 8 pencils cost \$4.40, then
4 pencils cost \$2.20,
2 pencils cost \$1.10, and
1 pencil costs \$0.55.

Therefore, 5 pencils cost

\$0.55 + \$2.20 = \$2.75.

Strategy 2: Find unit prices

First, find the cost of one pencil.

$$\frac{\$4.40}{8} = \$0.55$$

Then, multiply by 5 to find the cost of 5 pencils,

(\$0.55)(5) = \$2.75.

Sammy can crawl 12 feet in 3 seconds. At this rate, how far can Sammy crawl in $1\frac{1}{2}$ minutes?

Strategy 1: Make a table

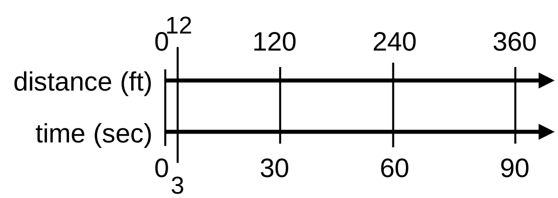
Distance	Time
12 ft	3 seconds
4 ft	1 second
240 ft	60 sec = 1 min
120 ft	30 sec = $\frac{1}{2}$ min
360 ft	90 sec = $1\frac{1}{2}$ min

Sammy can crawl 360 feet in $1\frac{1}{2}$ minutes.

Strategy 2: Make a Double Number Line

12 feet in 3 seconds is equivalent to
120 feet in 30 seconds

$1\frac{1}{2}$ minutes = 90 seconds.

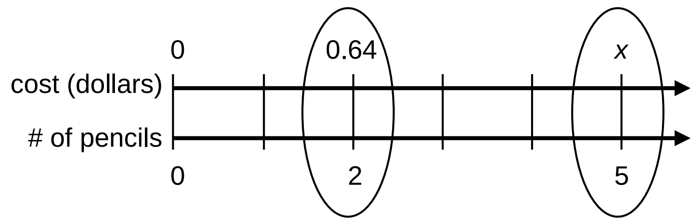


Sammy can crawl 360 feet in $1\frac{1}{2}$ minutes.

Writing Equations Based on Rates

Here are some ways to set up an equation to solve a rate problem. An equation in the form $\frac{a}{b} = \frac{c}{d}$ is commonly referred to as a “proportion.” Double number lines help make sense of this process. (See boxes on the next page for equation solving strategies.)

If 2 pencils cost $0.64, how much will 5 pencils cost?



Strategy 1: Compare rates (“between” two different units)

Create two rates from ratios that compare dollars to pencils. Equate expressions and solve for x.

$$\frac{x}{5} = \frac{0.64}{2}$$

x = 1.60 dollars for 5 pencils.

Note: The equation $\frac{5}{x} = \frac{2}{0.64}$ is another valid “between” equation for this problem.

Strategy 2: Compare like units (‘within” the same units)

Create one rate based on corresponding cost ratios and another rate based on the corresponding numbers of pencils ratios. Then, equate expressions and solve for x.

$$\frac{\text{cost}_{\text{case 1}}}{\text{cost}_{\text{case 2}}} = \frac{0.64}{x} \qquad \frac{\text{pencils}_{\text{case 1}}}{\text{pencils}_{\text{case 2}}} = \frac{2}{5}$$

$$\frac{0.64}{x} = \frac{2}{5}$$

x = 1.60 dollars for 5 pencils.

Note: The equation $\frac{x}{0.64} = \frac{5}{2}$ is another valid “within” equation for this problem.

Some Properties Relevant to Solving Equations
Here are some important properties of arithmetic and equality related to solving equations. • The <u>multiplication property of equality</u> states that equals multiplied by equals are equal. Thus, if $a = b$ and $c = d$, then $ac = bd$. Example: If $1 + 2 = 3$ and $5 = 9 - 4$, then $(1 + 2)(5) = 3(9 - 4)$. • The <u>cross-multiplication property for equations</u> states that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ $(b \neq 0, d \neq 0)$. This can be remembered with the diagram: $\frac{a}{b} \times \frac{c}{d}$. Example: If $\frac{5}{7} = \frac{12}{x}$, then $5 \bullet x = 7 \bullet 12$. To see that this property is reasonable, try simple numbers: If $\frac{3}{4} = \frac{6}{8}$, then $3 \bullet 8 = 4 \bullet 6$.

Applying Properties to Solve Proportion Equations	
Strategy 1: Multiplication Property of Equality	**Strategy 2: Cross-Multiplication Property**
Solve for x: $\frac{x}{12} = \frac{3}{8}$ (Multiplication Property of Equality) $(8 \bullet 12) \bullet \frac{x}{12} = \frac{3}{8} \bullet (8 \bullet 12)$ $8x = 36$ $x = \frac{36}{8}$ $x = 4\frac{1}{2}$	Solve for x: $\frac{x}{12} = \frac{3}{8}$ (Cross-multiplication property) $8 \bullet x = (3 \bullet 12)$ $8x = 36$ $x = \frac{36}{8}$ $x = 4\frac{1}{2}$

Simplifying Complex Fractions

Strategy 1: A complex fraction can be written as a division problem.

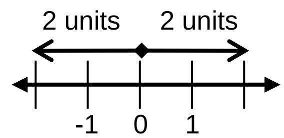
Example: $$\frac{\frac{1}{4}}{\frac{3}{8}} = \frac{1}{4} \div \frac{3}{8} = \frac{1}{4} \bullet \frac{8}{3} = \frac{8}{12} = \frac{2}{3}$$

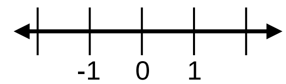
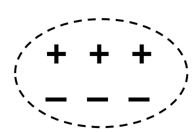
Strategy 2: A complex fraction can be multiplied by a form of the "big one" to create a denominator equal to one. Multiply the numerator and denominator each by the reciprocal of the denominator (in this case since the reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$). This process leaves a multiplication problem to compute.

Example: $$\frac{\frac{1}{4}}{\frac{3}{8}} \bullet \frac{\frac{8}{3}}{\frac{8}{3}} = \frac{\frac{1 \bullet 8}{4 \bullet 3}}{\frac{3 \bullet 8}{8 \bullet 3}} = \frac{\frac{8}{12}}{1} = \frac{8}{12} = \frac{2}{3}$$

While Strategy 2 seems to require more steps, this strategy makes more transparent the properties involved in writing the complex fraction in a more usable form.

STUDENT RESOURCES

Word or Phrase	Definition
absolute value	The absolute value $\lvert x \rvert$ of a number x is the distance from x to 0 on the number line. $\lvert 2 \rvert = 2$ and $\lvert -2 \rvert = 2$, because both 2 and -2 are 2 units from 0 on the number line. 
addend	See sum.
additive identity property	The additive identity property states that $a + 0 = 0 + a = a$ for any number a. In other words, the sum of a number and 0 is the number. We say that 0 is an additive identity. The additive identity property is sometimes called the addition property of zero. 3 + 0 = 3, 0 + 7 = 7, -5 + 0 = -5 = 0 + (-5)
additive inverse	The additive inverse of a is the number b such that $a + b = b + a = 0$. The additive inverse of a is denoted by $-a$. -4 is the additive inverse of 4.
additive inverse property	The additive inverse property states that $a + (-a) = 0$ for any number a. In other words, the sum of a number and its opposite is 0. The number $-a$ is the additive inverse of a. 3 + (-3) = 0, -5 + 5 = 0
difference	In a subtraction problem, the difference is the result of subtraction. The minuend is the number from which another number is being subtracted, and the subtrahend is the number that is being subtracted. 12 – 4 = 8 minuend subtrahend difference
integers	The integers are the whole numbers and their opposites. They are the numbers 0, 1, 2, 3, … and -1, -2, -3, … .
minuend	See difference.
negative numbers	Negative numbers are numbers that are less than zero, written $a < 0$. The negative numbers are the numbers to the left of 0 on a horizontal number line, or below zero on a vertical number line. The numbers -2, -4.76, and $-\frac{1}{4}$ are negative. The numbers 2 and 5.3, and 0 are NOT negative.

Word or Phrase	Definition
opposite of a number	The <u>opposite of a number</u> *n*, written $-n$, is its additive inverse. Algebraically, the sum of a number and its opposite is zero. Geometrically, the opposite of a number is the number on the other side of zero at the same distance from zero. The opposite of 1 is -1, because $1 + (-1) = -1 + 1 = 0$. The opposite of -1 is $-(-1) = 1$. Thus, the opposite of a number does not have to be negative. 
positive numbers	<u>Positive numbers</u> are numbers that are greater than zero, written $a > 0$. The positive numbers are the numbers to the right of 0 on a number line, or above zero on a vertical number line. The numbers 3, 2.6, and $\frac{3}{7}$ are positive. The numbers -3, -2.6, $-\frac{3}{7}$, and 0 are NOT positive.
rational numbers	<u>Rational number</u> are numbers expressible in the form $\frac{m}{n}$, where *m* and *n* are integers, and $n \neq 0$. $\frac{3}{5}$ is rational because it is a quotient of integers. $2\frac{1}{3}$ and 0.7 are rational numbers because they **can be** expressed as quotients of integers, namely $\frac{7}{3}$ and $\frac{7}{10}$, respectively. $\sqrt{2}$ and π are NOT rational numbers. They cannot be expressed as a quotient of integers.
subtrahend	See <u>difference</u>.
sum	A <u>sum</u> is the result of addition. In an addition problem, the numbers to be added are <u>addends</u>. 7 (addend) + 5 (addend) = 12 (sum)
whole numbers	The <u>whole numbers</u> are the natural numbers together with 0. They are the numbers 0, 1, 2, 3, … .
zero pair	In the counter model, a positive and a negative counter together form a <u>zero pair</u>. Let **+** represent a positive counter and let **–** represent a negative counter.  Then the figure to the right is an example of a collection of (three) zero pairs.

Mr. Mortimer's Magic Cubes

Mr. Mortimer discovered an amazing way to control the temperature of liquid. He invented magic hot and cold cubes to change the liquid's temperature. These magic cubes never melt or change in any way. For example, ice cubes melt, but magic cold cubes do not.

Hot Cubes (the basics):

- If you add 1 hot cube to a liquid, the liquid heats up by 1 degree.
- If you remove 1 hot cube from the liquid, the liquid cools down by 1 degree.

Cold Cubes (the basics):

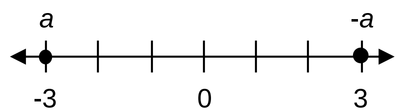
- If you add 1 cold cube to the liquid, the liquid cools down by 1 degree.
- If you remove 1 cold cube from the liquid, the liquid heats up by 1 degree.

How this temperature change model works		**For 1 cube**
Hot Cubes Positive (+)	Put in Heat → Hotter	add (+1) → +(+1) = +1
	Remove Heat → Colder	subtract (+1) → -(+1) = -1
Cold Cubes Negative (–)	Put in Cold → Colder	add (-1) → +(-1) = -1
	Remove Cold → Hotter	subtract (-1) → -(-1) = +1

Here are a few examples to show temperature change using magic hot and cold cubes.

	Simplest ways:		**Other Ways:**	
+4 degrees	Put in 4 hot cubes +(+4) = 4	Remove 4 cold cubes -(-4) = 4	Put in 6 hot cubes and put in 2 cold cubes +(+6) + (-2) = 4	Remove 6 cold cubes and remove 2 hot cubes -(-6) – (+2) = 4
-2 degrees	Remove 2 hot cubes -(+2) = -2	Put in 2 cold cubes +(-2) = -2	Remove 3 hot cubes and remove 1 cold cube -(+3) – (-1) = -2	Put in 3 cold cubes and put in 1 hot cube +(-3) + (+1) = -2
0 degrees	Do nothing 0		Put in 4 hot cubes and put in 4 cold cubes +(+4) + (-4) = 0	Remove 3 hot cubes and remove 3 cold cubes -(+3) – (-3) = 0

Representing the Additive Inverse

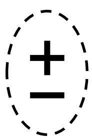
The minus sign may be used to show additive inverses. The identity $a + (-a) = 0$ means that $-a$ is the additive inverse of a. It is what we add to a to get 0.	Example: If $a = -3$, then $-a = 3$ The statement, "If a is equal to minus 3, then minus a is equal to 3" can be read: • If a is equal to the opposite of 3, then the opposite of a is equal to 3. When we add -3 and 3, the result is 0. 

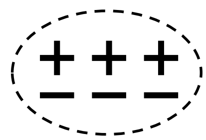
A Counter Model

This counter model is used to model integers.

Let + represent a positive counter with a value of positive 1
Let – represent a negative counter with a value of negative 1.

A zero pair is a pair with one positive counter and one negative counter.
Both representations below have a value of zero.

one zero pair: 

three zero pairs: 

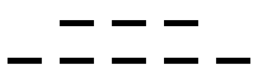
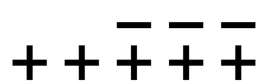
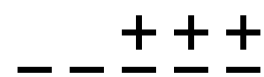
Below are some counter diagrams that represent the given integers:

	+4	**-2**	**0**
Simplest representation:	+ + + +	– –	(no counters)
Other representations:	+ + + + + –	– – + + – –	+ –
	+ + + + + + + – – –	– – – – – – + + + +	+ + + + – – – –

Counter Addition Sentence Frames

- Begin with a work space that has a value equal to 0.
- Build ___________ _____.
  positive/negative
- The plus (+) means to add.
- Add _____ ___________ counter(s).
  positive/negative
- The result is _____ ___________ counter(s).
  positive/negative

Integer Addition Using Counters

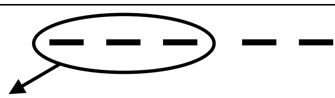
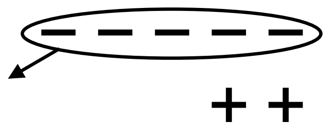
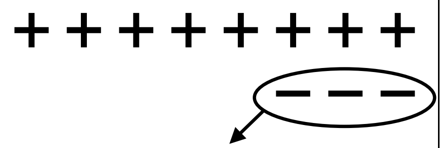
-3 + (-5) = -8	-3 + 5 = 2	3 + (-5) = -2
		
• Start with a work space equal to zero • Build negative 3 • The (+) means to add • Add 5 negative counters • The result is 8 negative counters	• Start with a work space equal to zero • Build negative 3 • The (+) means to add • Add 5 positive counters • The result is 2 positive counters	• Start with a work space equal to zero • Build positive 3 • The (+) means to add • Add 5 negative counters • The result is 2 negative counters

Rules for Addition of Integers

Rule 1: When the addends have the same sign, add the absolute values. Use the original sign in the answer.

Rule 2: When the addends have different signs, subtract the absolute values. Use the sign of the addend with the greatest absolute value in the answer.

Counter Subtraction Sentence Frames
• Begin with a work space that has a value equal to 0. • Build ___________ _____. positive/negative • The minus (–) means to subtract. • Subtract _____ ___________ counter(s). Introduce zero pairs if needed. positive/negative • The result is _____ ___________ counter(s). positive/negative

Integer Subtraction Using Counters		
-5 – (-3) = -2	-3 – (-5) = 2	5 – (-3) = 8
		
• Start with a work space equal to zero. • Build negative 5. • The (–) means to subtract. • Subtract 3 negative counters. I do not need zero pairs. • The result is 2 negative counters.	• Start with a work space equal to zero. • Build negative 3. • The (–) means to subtract. • Subtract 5 negative counters. I need zero pairs to do this. • The result is 2 positive counters.	• Start with a work space equal to zero. • Build positive 5. • The (–) means to subtract. • Subtract 3 negative counters. I need zero pairs to do this. • The result is 8 positive counters.

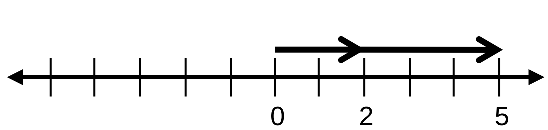
Rule for Subtraction of Integers
Rule: In symbols, $a - b = a + (-b)$ and $a - (-b) = a + b$. In words, the result is the same whether subtracting a quantity or adding its opposite. Examples: $6 - 4 = 6 + (-4) = 2$ $-3 - (-2) = -2 + 2 = -1$

Addition and Subtraction on a Number Line

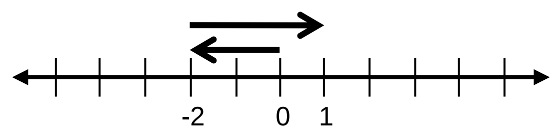
We can use arrows to represent addition and subtraction on a number line. For **adding** any two numbers:

- The absolute value of a number is represented by the arrow length.
- The first arrow begins at zero. If it's representing a positive number, the arrow points to the right. If it's representing a negative number, the arrow points to the left.
- If the second number is positive, the arrow points right. If the second number is negative, the arrow points left.
- The sum is represented by the end (tip) position of the second arrow.

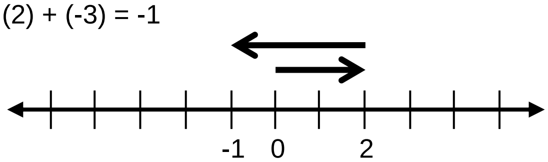
$(2) + (3) = 5$



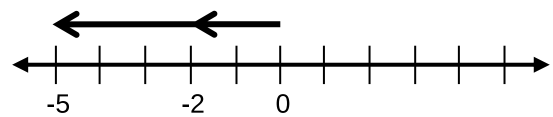
$(-2) + (3) = 1$



$(2) + (-3) = -1$



$(-2) + (-3) = -5$

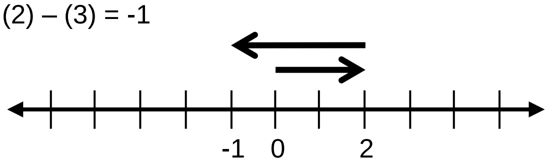


For **subtracting** any two numbers, remember that any minus sign signals doing the "opposite:"

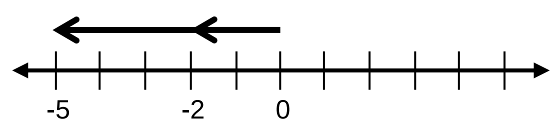
- The absolute value of a number is represented by the arrow length.
- The first arrow begins at zero. If it represents a positive number, the arrow points to the right. If it represents a negative number, the arrow points to the left.
- If the second number is positive, move the opposite of right (LEFT). If the second number is negative, move the opposite of left (RIGHT).
- The difference is represented by the end (tip) position of the second arrow.

Compare the subtraction problems below to the addition problems above. Notice that the first numbers and arrows are all identical to those above. Notice that the numbers subtracted are identical as well, and so the second arrows all point in the opposite direction.

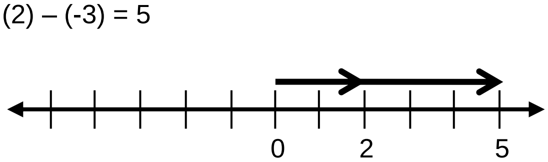
$(2) – (3) = -1$



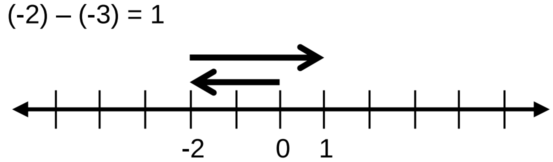
$(-2) – (3) = -5$



$(2) – (-3) = 5$



$(-2) – (-3) = 1$



STUDENT RESOURCES

Word or Phrase	Definition
distributive property	The <u>distributive property</u> states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c. $3(4 + 5) = 3(4) + 3(5)$ and $(4 + 5)8 = 4(8) + 5(8)$
exponential notation	The <u>exponential notation</u> b^n (read as " b to the <u>power</u> n ") is used to express n factors of b. The number b is the <u>base</u>, and the number n is the <u>exponent</u>. $2^3 = 2 \cdot 2 \cdot 2 = 8$. The base is 2 and the exponent is 3. $3^2 = 3 \cdot 3 = 9$. The base is 3 and the exponent is 2.
integers	The <u>integers</u> are the whole numbers and their opposites. They are the numbers 0, 1, 2, 3, … and -1, -2, -3, … .
inverse operation	The <u>inverse operation</u> to a mathematical operation reverses the effect of the operation. Addition and subtraction are inverse operations. Multiplication and division are inverse operations.
product	A <u>product</u> is the result of multiplying two or more numbers or expressions. The numbers or expressions being multiplied to form the product are <u>factors</u> of the product. 3 (factor) • 5 (factor) = 15 (product)
quotient	In a division problem, the <u>quotient</u> is the result of the division. 12 (dividend) ÷ 3 (divisor) = 4 (quotient)
rational numbers	<u>Rational numbers</u> are numbers expressible in the form $\frac{m}{n}$, where m and n are integers, and $n \neq 0$. $\frac{3}{5}$ is rational because it is a quotient of integers. $2\frac{1}{3}$ and 0.7 are rational numbers because they **can be** expressed as quotients of integers, namely $\frac{7}{3}$ and $\frac{7}{10}$, respectively. $\sqrt{2}$ and π are NOT rational numbers. They cannot be expressed as a quotient of integers. $\frac{7}{0}$ is undefined. It is NOT a rational number.

Symbols for Multiplication
The product of 8 and 4 can be written as:
8 times 4 8×4 $8 \bullet 4$ (8)(4) $\begin{array}{r} 8 \\ \underline{\times\ 4} \end{array}$
The product of 8 and the variable x is written simply as $8x$. We are cautious about using certain symbols for multiplication. The $\times$ could be misinterpreted as the variable x and the $\bullet$ could be misinterpreted as a decimal point.

Symbols for Division
The quotient of 8 and 4 can be written as:
8 divided by 4 $8 \div 4$ $4\overline{)8}$ $\frac{8}{4}$ 8/4
In algebra, the preferred way to show division is with fraction notation.

Mr. Mortimer's Magic Hot and Cold Cubes for Multiplication

Mr. Mortimer discovered an amazing way to control the temperature of liquid. He invented magic hot and cold cubes to change the liquid's temperature. These magic cubes never melt or change in any way. For example, ice cubes melt, but magic cold cubes do not.

Hot Cubes (the basics):

- If you add 1 hot cube to a liquid, the liquid heats up by 1 degree.
- If you remove 1 hot cube from the liquid, the liquid cools down by 1 degree.

For multiplication:

- If you put in packs of hot cubes to a liquid, the liquid heats up.
  For example, adding 2 packs of 10 hot cubes is like adding $2 \bullet 10 = 20$ hot cubes.
  The liquid heats up by 20 degrees.
- If you take out packs of hot cubes from a liquid, the liquid cools down.
  For example, subtracting 2 packs of 10 hot cubes is like subtracting $2 \bullet 10 = 20$ hot cubes.
  The liquid cools down by 20 degrees.

Cold Cubes (the basics):

- If you add 1 cold cube to the liquid, the liquid cools down by 1 degree.
- If you remove 1 cold cube from the liquid, the liquid heats up by 1 degree.

For multiplication:

- If you put in packs of cold cubes to a liquid, the liquid cools down.
  For example, adding 2 packs of 10 cold cubes is like adding $2 \bullet 10 = 20$ cold cubes.
  The liquid cools down by 20 degrees.
- If you take out packs of cold cubes from a liquid, the liquid heats up.
  For example, subtracting 2 packs of 10 cold cubes is like subtracting $2 \bullet 10 = 20$ cold cubes.
  The liquid heats up by 20 degrees.

Counter Multiplication Sentence Frames

- Begin with a workspace that has a value equal to 0.
- **If the first factor is positive**, we will place _____ groups on the workspace.
  If the first factor is negative, we will remove _____ groups on the workspace.
- The second factor is ____________ (positive/negative), so each group will contain _____ ____________ (positive/negative) counter(s).
- Introduce _____ zero pairs to remove these groups (if needed).
- The result is _____ ____________ (positive/negative) counter(s).

Integer Multiplication Using Counters

2(4) = 8

\+ + + +
\+ + + +

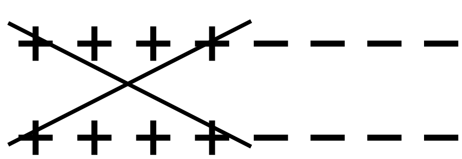
- Start with a work space equal to zero.
- The first factor is positive.
  We will put 2 groups on the workspace.
- The second factor is positive.
  Each group will contain 4 positive counters.
- [No zero pairs needed.]
- The result is 8 positive counters.

2(- 4) = - 8

− − − −
− − − −

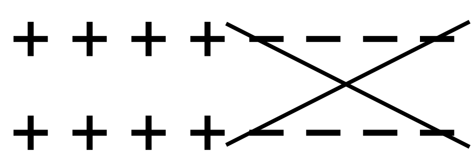
- Start with a work space equal to zero.
- The first factor is positive.
  We will put 2 groups on the workspace.
- The second factor is negative.
  Each group will contain 4 negative counters.
- [No zero pairs needed.]
- The result is 8 negative counters

- 2(4) = -8



- Start with a work space equal to zero.
- The first factor is negative.
  We will remove 2 groups from the workspace.
- The second factor is positive. Each group will contain 4 positive counters.
- Introduce at least 8 zero pairs.
- The result is 8 negative counters.

- 2(- 4) = 8



- Start with a work space equal to zero.
- The first factor is negative.
  We will remove 2 groups from the workspace.
- The second factor is negative. Each group will contain 4 negative counters.
- Introduce at least 8 zero pairs.
- The result is 8 positive counters.

Rules for Multiplication of Integers

Rule 1: The product of two numbers with the same sign is a positive number.

Think: (+)(+) = (+) and (–)(–) = (+)

Rule 2: The product of two numbers with opposite signs is a negative number.

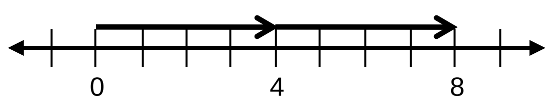
Think: (+)(–) = (–) and (–)(+) = (–)

Multiplication on a Number Line

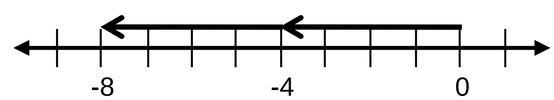
We can use arrows to represent multiplication on a number line. One interpretation for **multiplying** any two numbers is:

- The first factor tells us the number of arrows. The second factor tells us the length of each arrow.
- If the length of the arrow (second factor) is a positive number, then the arrow goes to the right. If the length of the arrow is a negative number, then the arrow goes to the left.
- If the number of arrows (first factor) is positive, then the number line diagram is complete. If the number of arrows is negative, then the entire diagram is reflected!

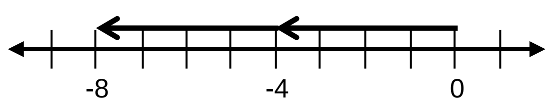
1. 2(4) = 8



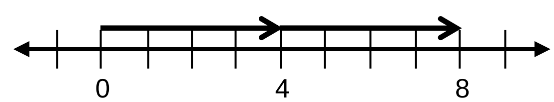
2. 2(-4) = -8



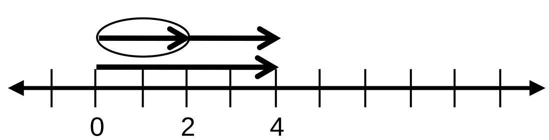
3. -2(4) = -8
(Example 1 reflected)



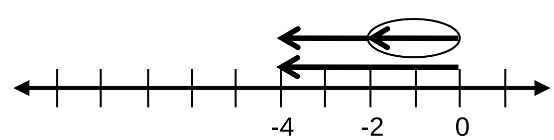
4. -2(-4) = 8
(Example 2 reflected)



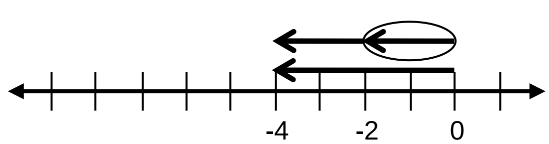
5. $\frac{1}{2}(4) = 2$



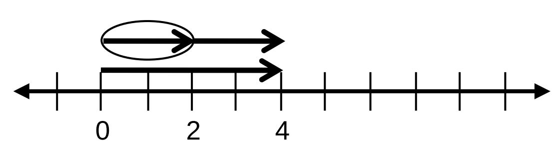
6. $\frac{1}{2}(-4) = -2$



7. $-\frac{1}{2}(4) = -2$
(Example 5 reflected)



8. $-\frac{1}{2}(-4) = 2$
(Example 6 reflected)



Rules for Division of Integers

Rule 1: The quotient of two numbers with the same sign is a positive number.

Think: $\frac{(+)}{(+)} = (+)$ and $\frac{(-)}{(-)} = (+)$

Rule 2: The quotient of two numbers with opposite signs is a negative number.

Think: $\frac{(+)}{(-)} = (-)$ and $\frac{(-)}{(+)} = (-)$

Mathematical Separators

Parentheses ( ) and square brackets [ ] are used in mathematical language as separators. The expression inside the parentheses or brackets is considered as a single unit. Operations are performed inside the parentheses before the expression inside the parentheses is combined with anything outside the parentheses.

$$5 - (2 + 1) = 5 - (3) = 2$$

In the example below, operate on the expression in the innermost separator first and work your way out.

$$20 \div [6 - (4 - 8)] = 20 \div [6 - (-4)] = 20 \div 10 = 2$$

The horizontal line used for a division problem is also a separator. It separates the expressions above and below the line, so the numerator and denominator must be simplified completely before dividing.

$$\frac{20 + 10}{5 \bullet 2} = \frac{30}{10} = 3$$

Order of Operations

There are many mathematical conventions that enable us to interpret mathematical notation and to communicate efficiently about common situations. The agreed-upon rules for interpreting mathematical notation, important for simplifying arithmetic and algebraic expressions, are called the order of operations.

1. Do the operations in grouping symbols first (e.g., use rules 2 – 4 inside parentheses).
2. Calculate all the expressions with exponents.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

$$\frac{11+(17-2\bullet 3^2)}{5} = \frac{11+(17-2\bullet 9)}{5} = \frac{11+(17-18)}{5} = \frac{11+(-1)}{5} = \frac{10}{5} = 2$$

There are many times for which these rules make complete sense and are quite natural. Take this case, for example:

You purchase 2 bottles of water for $1.50 each and 3 bags of peanuts for $1.25 each. Write an expression for this situation, and simplify the expression to find the total cost.

$$\underbrace{2\bullet(1.50)}_{3.00} + \underbrace{3\bullet(1.25)}_{3.75} = \$6.75$$

In this problem, it is natural to find the cost of the 2 bottles of water and then the cost of the 3 bags of peanuts prior to adding these amounts together. In other words, we perform the multiplication operations before the addition operation.

Note however that if we were to perform the operations in order from left to right (as we read the English language from left to right), we would obtain a different result:

WRONG → 2(1.50) = 3 → 3 + 3 = 6 → 6(1.25) = $7.50

Using Order of Operations to Simplify Expressions		
Order of Operations	**Example** $$\frac{40 - 2 \bullet 5^2 - (8 - 6)}{4 + 2 \bullet 10}$$	**Comments**
Simplify expressions within grouping symbols.	$\frac{40 - 2 \bullet 5^2 - 2}{4 + 2 \bullet 10}$	Parentheses are grouping symbols: (8 – 6) = 2 The fraction bar, used for division, is also a grouping symbol, so the numerator and denominator must be simplified completely prior to dividing.
Calculate all the expressions with exponents.	$\frac{40 - 2 \bullet 25 - 2}{4 + 2 \bullet 10}$	$5^2 = 5 \bullet 5 = 25$
Perform multiplication and division from left to right.	$\frac{40 - 50 - 2}{4 + 20}$	In the numerator: Multiply 2 • 25 = 50. In the denominator: Multiply 2 • 10 = 20.
Perform addition and subtraction from left to right.	$\frac{-12}{24}$	In the numerator: Subtract from left to right 40 – 50 – 2 = -12. In the denominator: Add 4 + 20 = 24
	$\frac{-1}{2}$ or $-\frac{1}{2}$	Now the groupings in both the numerator and denominator have been simplified, so the final division can be performed.

STUDENT RESOURCES

Word or Phrase	Definition
additive inverse property	The additive inverse property states that $a + (-a) = 0$ for any number a. In other words, the sum of a number and its opposite is 0. The number $-a$ is the additive inverse of a. $3 + (-3) = 0$, $-25 + 25 = 0$
coefficient	A coefficient is a number or constant factor in a term of an algebraic expression. In the expression $3x + 5$, 3 is the coefficient of the term $3x$, and 5 is the constant term.
constant term	A constant term in an algebraic expression is a term that has a fixed numerical value. In the expression $5 + 2x + 3$, the terms 5 and 3 are constant terms. If this expression is rewritten as $2x + 8$, the term 8 is the constant term of the new expression.
distributive property	The distributive property states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c. $3(4 + 5) = 3(4) + 3(5)$; $(4 + 5)8 = 4(8) + 5(8)$; $6(8 – 1) = 6(8) – 6(1)$
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
equivalent expressions	Two mathematical expressions are equivalent if, for any possible substitution of values for the variables, the two resulting numbers are equal. In particular, two numerical expressions are equivalent if they represent the same number. See expression. The numerical expressions $3 + 2$ and $6 – 1$ are equivalent. Both are equal to 5. The algebraic expressions $3(x + 2)$ and $3x + 6$ are equivalent. For any value of the variable x, the expressions represent the same number.
evaluate	Evaluate refers to finding a numerical value. To evaluate an expression, replace each variable in the expression with a value and then calculate the value of the expression. To evaluate the numerical expression $3 + 4(5)$, we calculate $3 + 4(5) = 3 + 20 = 23$. To evaluate the variable expression $2x + 5$ when $x = 10$, we calculate $2x + 5 = 2(10) + 5 = 20 + 5 = 25$.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8+x}{10}$, and $4v – w$.

<table>
<tr><th>Word or Phrase</th><th>Definition</th></tr>
<tr><td>input-output rule</td><td>An input-output rule for a sequence of values is a rule that establishes explicitly an output value for each given input value.

<table>
<tr><td>input value (x)</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>x</td></tr>
<tr><td>output value (y)</td><td>1.5</td><td>3</td><td>4.5</td><td>6</td><td>7.5</td><td>$1.5x$</td></tr>
</table>

In the table above, the input-output rule could be $y = 1.5x$. In other words, to get the output value, multiply the input value by 1.5. If $x = 100$, then $y = 1.5(100) = 150$.

The “independent variable” is typically associated with the input value, and the “dependent variable” is typically associated with the output value.</td></tr>
<tr><td>like terms</td><td>See terms.</td></tr>
<tr><td>proportional</td><td>Two variables are proportional if the values of one are the same constant multiple of the corresponding values of the other. The variables are said to be in a proportional relationship, and the constant is referred to as the constant of proportionality.

If Wrigley eats 3 cups of kibble each day, then the number of cups of kibble is proportional to the number of days. If x is the number of days, and y is the number of cups of kibble, then $y = 3x$. The constant of proportionality is 3.</td></tr>
<tr><td>proportional relationship</td><td>See proportional.</td></tr>
<tr><td>simplify</td><td>Simplify refers to converting a numerical or variable expression to a simpler form. A variable expression might be simplified by combining like terms. A fraction might be simplified by dividing numerator and denominator by a common divisor.

$2x + 6 + 5x + 3 = 7x + 9$ $\frac{8}{12} = \frac{2}{3}$</td></tr>
<tr><td>terms</td><td>The terms in a mathematical expression involving addition (or subtraction) are the quantities being added (or subtracted). Terms with the same variable part are called like terms.

The expression $2x + 6 + 3x + 5$ has four terms: $2x$, 6, $3x$, and 5. The terms $2x$ and $3x$ are like terms, since each is a constant multiple of x. The terms 6 and 5 are like terms, since each is a constant.</td></tr>
<tr><td>variable</td><td>A variable is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to quantities that vary in a relationship (as in a formula or an input-output rule). They may refer to unknown quantities in expressions, equations or inequalities. Finally, they may be used to generalize rules of arithmetic.

In the equation $d = rt$, the quantities d, r, and t are variables.

In the equation $2x = 10$, the variable x may be referred to as the unknown.

The equation $a + b = b + a$ generalizes the commutative property of addition for all numbers a and b.</td></tr>
</table>

The Coordinate Plane: Quadrant I

In this packet, all graphing is done in Quadrant I, because all coordinates graphed are nonnegative.
A coordinate plane is determined by a horizontal number line (the x-axis) and a vertical number line (the y-axis) intersecting at the zero on each line. The point of intersection (0, 0) of the two lines is called the origin. Points are located using ordered pairs (x, y).

- The first number (x-coordinate) indicates how far the point is to the right of the y-axis.
- The second number (y-coordinate) indicates how far the point is above the x-axis.

Point, coordinates, and interpretation

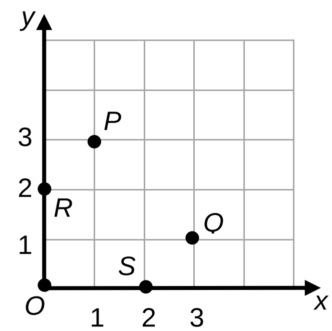
O (0, 0) → at the intersection of the axes

P (1, 3) → start at the origin, move 1 unit right, then 3 units up

Q (3, 1) → start at the origin, move 3 units right, then 1 unit up

R (0, 2) → start at the origin, move 0 units right, then 2 units up

S (2, 0) → start at the origin, move 2 units right, then 0 units up



Multiple Representations: Tables, Graphs, and Equations

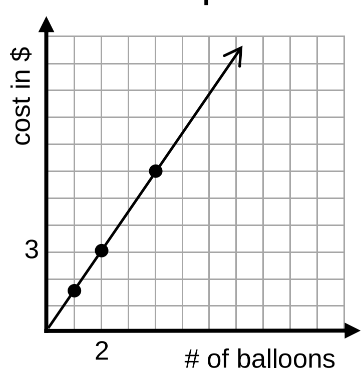
Suppose 4 balloons cost $6.00 and each balloon is the same price. Here are some representations for this relationship.

Table

Number of Balloons	Cost in $
4	6.00
2	3.00
1	1.50
8	12.00

Note that the unit price is $1.50 per balloon.

Graph



Numbers of balloons must be discrete values (specifically, whole numbers). A trend line may be drawn to show a growth pattern.

Equation (input-output rule)

Let y = cost in dollars
Let x = number of balloons.

We can see from the table that the unit price is 1.50 dollars per balloon.

It appears that multiplying any input value by 1.5 yields its corresponding output value.

Therefore, $y = 1.5x$.

Variables in Algebra	
Loosely speaking, variables are quantities that can vary. Variables are represented by letters or symbols. Variables have many different uses in mathematics. The use of variables, together with the rules of arithmetic, makes algebra a powerful tool. Three important ways that variables appear in algebra:	
Usage	**Examples**
Variables can represent an *unknown quantity* in an equation or inequality. In this case, the equation is valid only for specific value(s) of the variable.	$x + 4 = 9$ $5n = 20$ $y < 6$
Variables can represent *quantities that vary* in a relationship. In this case, there is always more than one variable in the equation.	Formula: $P = 2\ell + 2w$, $A = s^2$ Function (input-output rule): $y = 5x$, $y = x + 3$
Variables can represent *quantities in statements that generalize* rules of arithmetic. In this case, there may be one or more variables.	Commutative property of addition: $x + y = y + x$ Distributive property: $x(y + z) = xy + xz$

Evaluate or Simplify?

We use the word "evaluate" when we want to calculate the value of an expression.

To evaluate 16 – 4(2), follow the rules for order of operations and compute.

$$16 - 4(2) = 16 - 8 = 8$$

To evaluate $6 + 3x$ when $x = 2$, substitute 2 for x and calculate.

$$6 + 3(2) = 6 + 6 = 12$$

We use the word "simplify" when rewriting a number or an expression in a form more easily readable or understandable.

To simplify $2x + 3 + 5x$, combine like terms: $2x + 3 + 5x = 7x + 3$.

Sometimes it may not be clear what is the simplest form of an expression. For instance, by the distributive property, $4(x + 2) = 4x + 8$. For some applications, $4(x + 2)$ may be considered simpler than $4x + 8$, but for other applications, $4x + 8$ may be considered simpler than $4(x + 2)$.

Equivalent Expressions

Two numerical expressions are <u>equivalent</u> if they are equal.

$2 + 4$ and $-2 + 8$ are equivalent numerical expressions. They are both equal to 6.

Two mathematical expressions are <u>equivalent</u> if, for any possible substitution of values for the variables, the two resulting values are equal.

The expressions $x + 2x$ and $4x – x$ are equivalent. For any value of the variable x, the expressions represent the same number. We see this by combining like terms.

$$x + 2x = 3x \text{ and } 4x – x = 3x$$

The expressions x^2 and $2x$ are NOT equivalent. While they happen to be equal if $x = 0$ or $x = 2$, they are not equal for all possible values of x. For instance, if $x = 1$, then $x^2 = 1$ and $2x = 2$.

Properties of arithmetic, such as the distributive property, can be used to write expressions in different, equivalent ways.

$$4x + 6x = (4 + 6)x \qquad 24x + 9x = 3(8x + 3x) = 3x(8 + 3)$$

Simplifying Expressions Using a Model

In mathematics, we <u>simplify</u> a numerical or algebraic expression by rewriting it in a less complicated form.

We can illustrate simplifying expressions using a cups and counters model.

Positive Counter	Negative Counter	Cup	Upside-down Cup
draw as: **+** value: +1	draw as: **–** value: -1	draw as: **V** value: unknown (x)	draw as: **Λ** value: unknown ($-x$)

Expressions	Pictures	Descriptions
$2(x + 3)$ $= 2x + 6$	V + + + V + + +	Build the expression. Think: 2 groups of $x + 3$, which is an application of the distributive property.
$-2(x + 3)$ $= -2x – 6$	Λ – – – Λ – – –	Build the expression. Think: 2 groups of $x + 3$ from above -> then build the opposite (distributive property).
$-2(x – 3)$ $= -2x + 6$	Λ + + + Λ + + +	Build the expression. Think: 2 groups of $x – 3$ -> then build the opposite (distributive property).
$4x – 2(x – 3)$ $= 4x – 2x + 6$ $= 2x + 6$	V (V Λ) + + + V (V Λ) + + +	Build the expression. Think: 4x AND 2 groups of $x – 3$ -> then build the opposite of the groups (distributive property). Then combine like terms (think zero pairs).

STUDENT RESOURCES

Word or Phrase	Definition
boundary point of a solution set	A <u>boundary point of a solution set</u> is a point for which any interval surrounding the point on the number line contains both solutions and non-solutions. If the solution set is an interval, the boundary points of the solution set are the endpoints of the interval. The boundary point for BOTH $x < -1$ AND *x* £ -1 is $x = -1$. In the first case the boundary point IS NOT part of the solution set (open circle). In the second case it IS (closed circle).  
equation	An <u>equation</u> is a mathematical statement that asserts the equality of two expressions.
expression	A mathematical <u>expression</u> is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are $7x$, $a + b$, $4v - w$, $\frac{8 + x}{10}$, and 19.
inequality	An <u>inequality</u> is a mathematical statement that asserts the relative size or order of two objects. When the expressions involve variables, a <u>solution to the inequality</u> consists of values for the variables which, when substituted, make the inequality true. $5 > 3$ is an inequality. $x + 3 > 4$ is an inequality. All values for *x* that are greater than 1 are solutions to this inequality.
solution to an equation	A <u>solution to an equation</u> involving variables consists of values for the variables which, when substituted, make the equation true. The value $x = 8$ is a solution to the equation $10 + x = 18$. If we substitute 8 for *x* in the equation, the equation becomes true: $10 + 8 = 18$.
solve an equation	To <u>solve an equation</u> refers to finding all values for the variables in the equation that, when substituted, make the equation true. Values that make an equation true are called solutions to the equation. To solve the equation $2x = 6$, one might think “two times what number is equal to 6?” Since $2(3) = 6$, the only value for *x* that satisfies this condition is 3. Therefore 3 is the solution.
substitution	<u>Substitution</u> refers to replacing a value or quantity with an equivalent value or quantity. If $x + y = 10$, and $y = 8$, then we may substitute this value for *y* in the equation to get $x + 8 = 10$.

Properties of Arithmetic

Properties of arithmetic govern the manipulation of expressions (mathematical phrases).
For any three numbers *a*, *b*, and *c*:

- ✓ Associative property of addition
  $a + (b + c) = (a + b) + c$
- ✓ Commutative property of addition
  $a + b = b + a$
- ✓ Additive identity property
  (addition property of 0)
  $a + 0 = 0 + a = a$
- ✓ Additive inverse property
  $a + (-a) = -a + a = 0$
- ✓ Associative property of multiplication
  $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
- ✓ Commutative property of multiplication
  $a \bullet b = b \bullet a$
- ✓ Multiplicative identity property
  (multiplication property of 1)
  $a \bullet 1 = 1 \bullet a = a$
- ✓ Multiplicative inverse property
  $a \bullet \frac{1}{a} = \frac{1}{a} \bullet a = 1$
- ✓ Distributive property relating addition and multiplication
  $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers *a*, *b*, and *c*.

Properties of Equality

Properties of equality govern the manipulation of equations (mathematical sentences).

For any three numbers *a*, *b*, and *c*:

- ✓ Addition property of equality
  (Subtraction property of equality)
  If $a = b$ and $c = d$, then $a + c = b + d$.
- ✓ Multiplication property of equality
  (Division property of equality)
  If $a = b$ and $c = d$, then $ac = bd$
- ✓ Reflexive property of equality: $a = a$
- ✓ Symmetric property of equality:
  If $a = b$, then $b = a$
- ✓ Transitive property of equality:
  If $a = b$, and $b = c$, then $a = c$

Solving Equations Using a Substitution Strategy

Method 1: To solve an equation using substitution, apply your knowledge of arithmetic facts to find values that make the equation true.

Example 1: Solve $-3x = 15$.

- Think: ***What number times -3 is 15?***
- Since -3(-5) = 15, x = -5.

Example 2: Solve $12 = 20 – k$.

- Think: ***20 minus what equals 12?***
- Since 20 – 8 = 12, k = 8.

Method 2: Use the “cover-up” method and proceed as above.

Example 3: Solve $\frac{n + 20}{3} = 8$

- Cover up $n + 20$ →  over 3 = 8
- Think: ***What divided by 3 equals 8?***
- Since $\frac{24}{3} = 8$, you are covering up 24
- Think: ***What plus 20 equals 24?***
- Since 4 + 20 = 24, n = 4

Example 4: Solve $5(m – 2) = -20$.

Cover up $m – 2$ → 5 () = -20

- Think: ***5 times what equals -20?***
- Since 5(-4) = -20, you are covering up -4
- Think: ***What minus 2 equals -4?***
- Since -2 – 2 = -4, m = -2

Similar Phrases with Different Meanings

Sometimes it is useful to “translate” a string of words into symbols.

String of Words	Example	Symbols	Classification
is less than	4 is less than 10	$4 < 10$	inequality
less than	4 less than 10	10 – 4	expression
is greater than	7 is greater than 2 + 3	$7 > 2 + 3$	inequality
greater than	7 greater than 2 + 3	(2 + 3) + 7	expression

“Is greater than” includes the word “is.” Therefore, it behaves like a mathematical verb. This string of words is used to make a mathematical sentence (an inequality in this case).

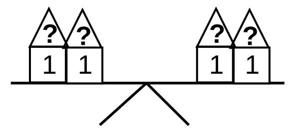
“Greater than” is a string of words without a verb. It translates into an expression. In English, we connect phrases with verbs to make sentences. The same is true in mathematics.

Balance Scales and Laws of Equality

Balance scales are physical representations of equations because both sides of a balanced scale must have the same weight, and both sides of an equation must have the same value.

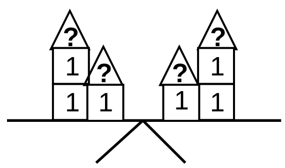
We imagine that each [1] represents one unit of weight and each [?] represents an unknown weight (not equal to zero). To represent unknowns, a popular variable is *x*.

Start with a balanced scale like the one to the right, which represents the equation $2x + 2 = 2x + 2$.



Example 1: Add the same thing to both sides, like 1.

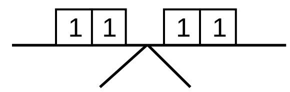
New scale:
(still balanced)



New equation: $2x + 2 + 1 = 2x + 2 + 1$
$2x + 3 = 2x + 3$

Example 2: Subtract the same thing from both sides, like 2*x*.

New scale:
(still balanced)

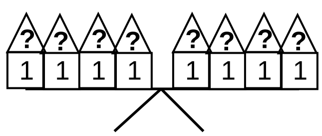


New equation: $2x + 2 – 2x = 2x + 2 – 2x$
$2 = 2$

In examples 1 and 2 the <u>addition property of equality</u> is applied (see Properties of Equality). Note that this property extends to subtraction as well. (See example 2 above).

Example 3: Multiply both sides by the same thing, like 2.

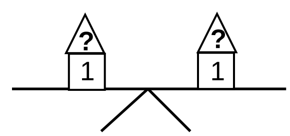
New scale:
(still balanced)



New equation: $2(2x + 2) = 2(2x + 2)$
$4x + 4 = 4x + 4$

Example 4: Divide both sides by the same thing, like 2.

New scale:
(still balanced)



New equation: $\frac{2x+2}{2} = \frac{2x+2}{2}$
$x + 1 = x + 1$

In examples 3 and 4 the <u>multiplication property of equality</u> is applied (see Properties of Equality). Note that this property extends to division as well.

Solving Equations Using a Model

Let **+** represent 1 Let **V** represent the unknown (like x)

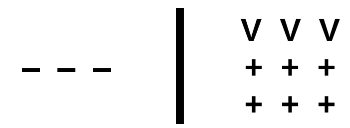
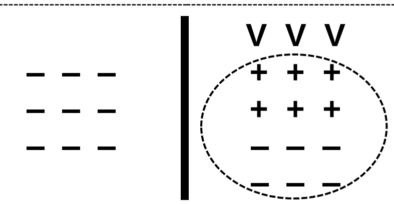
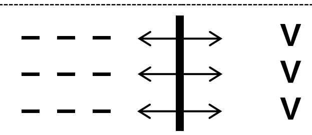
Let **–** represent -1 Let **Λ** represent the opposite of the unknown (like $-x$)

As you solve equations, think:

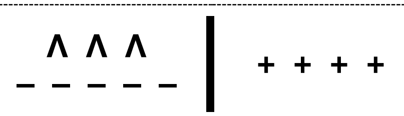
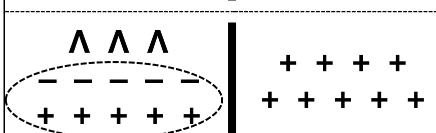
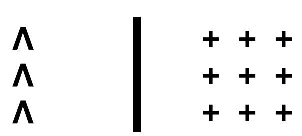
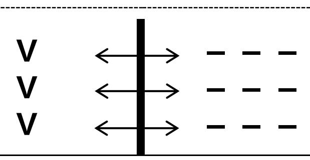
- Can I simplify one or both sides? That is, focus on what can be done to each expression alone.
- What can I do to both sides? That is, focus on what can be done to the equation.

The following examples illustrate one solution path. Others paths are possible to arrive at the same solutions.

Example 1: Solve $-3 = 3(x + 2)$.
Check (after solving): $3(-3) + 6 \rightarrow -9 + 6 \rightarrow -3 = -3$

Picture	Equation	Comments
	$-3 = 3x + 6$	build the equation (think: 3 groups of x + 2) and rewrite
	$-3 = 3x + 6$ $+(-6) = +(-6)$ $-9 = 3x$	add -6 to both sides and remove zero pairs
	$\frac{-9}{3} = \frac{3x}{3}$ $-3 = x$	divide both sides by 3 to put counters equally into cups

Example 2: Solve: $-2x - 5 - x = 4$
Check (after solving): $-2(-3) - 5 - (-3) \rightarrow 6 - 5 + 3 \rightarrow 1 + 3 = -3 + 7 \rightarrow 4 = 4$

Picture	Equation	Comments
	$-3x - 5 = 4$	build the equation and rewrite (collect like terms)
	$-3x - 5 = 4$ $+5 = +5$ $-3x = 9$	add 5 to both sides and remove zero pairs
	$(-1)(-3x) = (-1)(9) \rightarrow 3x = -9$ $\frac{3x}{3} = \frac{-9}{3}$ $x = -3$	take the opposite of both sides (accomplished by multiplying both sides by -1), and then divide both sides by 3
	$\frac{-3x}{-3} = \frac{9}{-3}$ $x = -3$	The above actions have the same effect as diving by -3 in a single step

Using Algebraic Techniques to Solve Equations

To solve equations using algebra:

- Use the properties of arithmetic to simplify each side of the equation (e.g., associative properties, commutative properties, inverse properties, distributive property).
- Use the properties of equality to isolate the variable (e.g., addition property of equality, multiplication property of equality).

- Solve: $-2 - 3 = 5x - 2x + 7$ for x

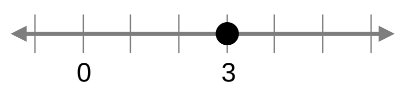
Equation	Comments
$-5 = 3x + 7$	• Collect like terms ($-2 - 3 = -5$; $5x - 2x = 3x$). Note that this is an application of the distributive property because $(5 - 2)x = 3(x)$.
$-5 = 3x + 7$ $\underline{-7} \quad \underline{-7}$ $-12 = 3x$	• Addition property of equality (subtract 7 from both sides) • Additive inverse property ($7 + (-7) = 0$)
$\frac{-12}{3} = \frac{3x}{3}$ $-4 = x$	• Multiplication property of equality (divide both sides by 3 or multiply both sides by $\frac{1}{3}$) • Multiplicative identity property ($1x = x$)

Graphing Inequalities

When graphing solutions to inequalities on the number line, we will use arrows to represent sets of solutions that extend indefinitely in one direction or the other. These arrows should not be confused with the arrows used to denote distance and direction above number ines in Units 3 and 4.

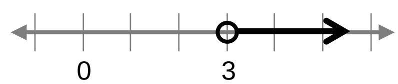
- Integers are graphed as closed circles.

  Example: $x = 3$



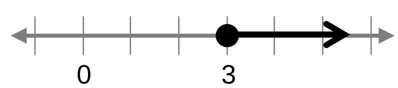
- Solutions to inequalities that involve the statements "is less than" (<) and "is greater than" (>) are graphed with an open circle, indicating that the boundary number *is not* included in the solution set.

  Example: $x > 3$



- Solutions to inequalities that involve the statements "is less than or equal to" (≤) and "is greater than or equal to" (≥) are graphed with a closed circle, indicating that the boundary number *is* included in the solution set.

  Example: $x \geq 3$



Reversing the Direction of an Inequality

When multiplying or dividing both sides of an inequality by a negative number, the direction of the inequality reverses.

Original inequality	Do to both sides	Resulting inequality	Direction reverses?
$10 > -4$	Add 2	$12 > -2$	No
	Subtract 2	$8 > -6$	No
	Multiply by 2	$20 > -8$	No
	Divide by 2	$5 > -2$	No
$-10 < 4$	Add -2	$-12 < 2$	No
	Subtract -2	$-8 < 6$	No
	Multiply by -2	$20 > -8$	**Yes**
	Divide by -2	$5 > -2$	**Yes**

The direction of an inequality reverses **ONLY** when multiplying or dividing both sides of an inequality by a negative number.

Note that it does not matter if there are negative numbers in the original inequality or not.

Solving Inequalities in One Variable

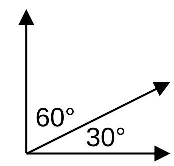
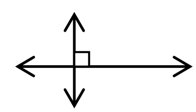
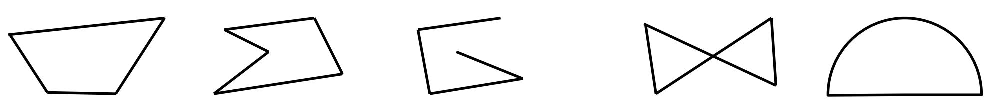
When solving a linear inequality, treat the inequality as if it were an equation. When multiplying or dividing both sides of the inequality by a negative number, reverse the direction of the inequality.

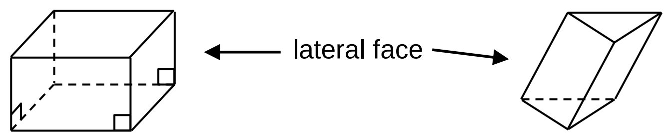
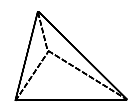
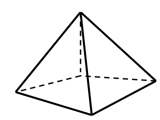
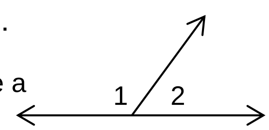
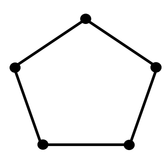
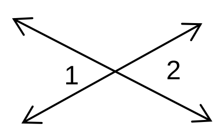
Example 1	Comments
$-4x + 1 \leq 13$ $-1 \quad -1$	(Subtraction) Do not reverse the inequality.
$-4x \leq 12$ $\frac{-4x}{-4} \geq \frac{12}{-4}$	(Division by a negative number) **Reverse the inequality symbol.**
$x \geq -3$	Solutions

Example 2	Comments
$2x - 9 \leq -13$ $+9 \quad +9$	(Addition) Do not reverse the inequality.
$2x \leq -4$ $\frac{2x}{2} \leq \frac{-4}{2}$	(Division by a positive number) **Do not reverse the inequality symbol.**
$x \leq -2$	Solutions

Error alert. The inequality does not always reverse when solving an inequality that includes negatives. Example 2 illustrates this.

STUDENT RESOURCES

Word or Phrase	Definition
adjacent angles	Two angles are adjacent if they have the same vertex and share a common ray, and they lie on opposite sides of the common ray. $\angle ABC$ and $\angle CBD$ are adjacent angles.
complementary angles	Two angles are complementary if the sum of their measures is 90°. Two angles that measure 30° and 60° are complementary. 
complementary angles	Two angles are complementary if the sum of their measures is 90°. Two angles that measure 30° and 60° are complementary.
cross section	The intersection of a solid figure with a plane is a cross section of the figure.
parallel	Two lines in a plane are parallel if they do not meet. Two line segments in a plane are parallel if the lines they lie on are parallel.
perpendicular	Two lines are perpendicular if they intersect at right angles. 
plane	A plane refers to a flat two-dimensional surface that has no holes and that extends to infinity in all directions.
polygon	A polygon is a special kind of figure in a plane made up of a chain of line segments laid end-to-end to enclose a region. Each endpoint of a segment of the polygon meets one other segment, otherwise the segments do not meet each other. The line segments are the sides (or edges) of the polygon, and the endpoints of the line segments are the vertices of the polygon. A polygon divides the plane into two regions, an "inside" and an "outside." The region inside a polygon may also be referred to as a polygon.  polygons not polygons

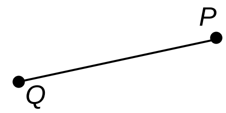
Word or Phrase	Definition
prism	A prism is a solid figure in which two faces (the bases) are identical parallel polygons, and the other faces (referred to as the lateral faces) are parallelograms. If the lateral faces are perpendicular to the bases, the prism is a right prism. Otherwise, the prism is an oblique prism.  A right rectangular prism is a right prism whose bases are rectangles and faces are rectangles. An oblique triangular prism is a prism whose bases are triangles and faces are parallelograms.
pyramid	A pyramid is a solid figure in which one face (the base) is a polygon, and the other faces are triangles with a common vertex (the apex). Each edge of the base is the side of a triangular face with the opposite vertex at the apex. A triangular pyramid is a pyramid with a triangular base.  A square pyramid is a pyramid with a square base. The Egyptian pyramids are examples of square pyramids. 
solid figure	A solid figure refers to a figure in three-dimensional space such as a prism or a cylinder.
supplementary angles	Two angles are supplementary if the sum of their measures is 180°. Angles 1 and 2 are supplementary because they determine a straight line, or 180°. 
vertex	A vertex (pl. vertices) of a polygon or solid figure is a point where two edges meet. A pentagon has five vertices. 
vertical angles	Two angles are vertical angles if they are opposite angles formed by a pair of intersecting lines. $\angle 1$ and $\angle 2$ are vertical angles. 

Symbols and Conventions for Geometry Notation

Below are some geometry notations we will use. Note that we use absolute values to denote lengths of segments and measures of angles. This is consistent with more advanced levels of mathematics.

<u>Points</u> are named by capital letters.

The <u>line segment</u> from P to Q is denoted by $\overline{PQ}$.



The <u>length of the line segment</u> from P to Q is denoted by $|PQ|$, which is shorthand for $|\overline{PQ}|$.

The symbol for <u>triangle</u> is $\triangle$.

- The triangle in Figure 1 below may be denoted by $\triangle LMN$, or also by $\triangle LNM$. Vertices may be listed in either a clockwise or counterclockwise direction starting from any of the three vertices.

The symbol for <u>angle</u> is $\angle$.

- The angle at the top of Figure 1 below can be denoted by $\angle NLM$, or by $\angle a$ or by $\angle L$.
- The pair of adjacent angles in Figure 2 below are $\angle FGJ$ and $\angle HGF$. They share the common <u>ray</u> $\overrightarrow{GF}$. The two adjacent angles together form the angle $\angle JGH$.

Error alert: Using "$\angle G$" to name the angle below is ambiguous. We do not know if it refers to $\angle JGF$, $\angle FGH$, or $\angle JGH$.

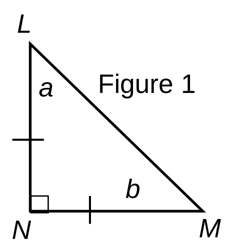
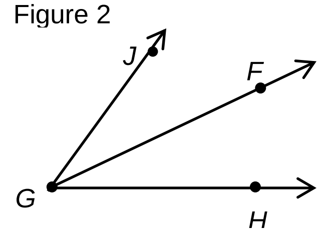


Figure 1

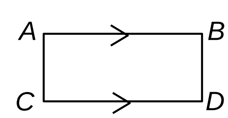
Figure 2



The <u>measure of an angle</u> $\angle N$ is denoted by $|\angle N|$. The small square at N indicates that $\angle LNM$ is a right angle, that is, that $|\angle LNM| = 90°$.

The single hash marks on the segments $\overline{LN}$ and $\overline{NM}$ indicate that the segments have equal length, that is, $|LN| = |NM|$.

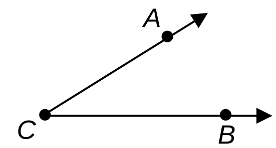
The arrow marks on the segments $\overline{AB}$ and $\overline{CD}$ indicate that the segments are parallel.



Classifying Angles by their Degree Measure

An angle is a geometric shape formed by two (distinct) rays that share a common endpoint (the vertex of the angle).

The angle in the figure to the right can be named any one of the following:

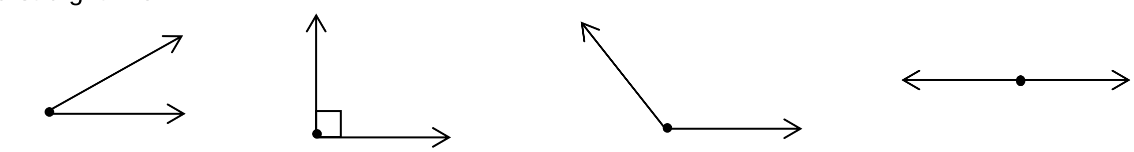


$\angle ACB$ *or* $\angle BCA$ *or* $\angle C$

The point C is the vertex of the angle. The rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$ meet at C and form the sides of the angle.

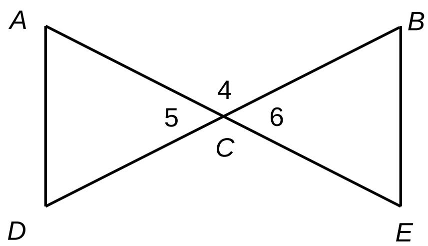
To each angle is assigned a degree measure between 0 and 180 degrees, which indicates the size of the angle. Angles may be classified by their degree measure.

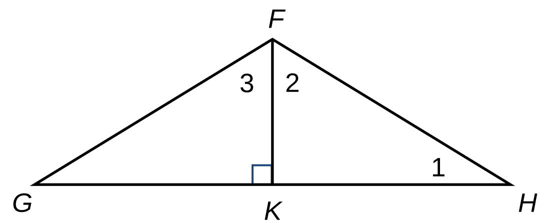
- An acute angle is an angle whose measure is less than 90°.
- A right angle is an angle whose measure is exactly 90°.
- An obtuse angle is an angle whose measure is between 90° and 180°.
- A straight angle is an angle whose measure is 180°. The sides of a straight angle are opposite rays that form a straight line.



acute angle right angle obtuse angle straight angle

Special Angle Pairs



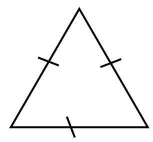
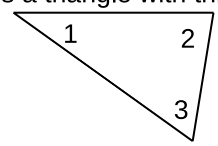


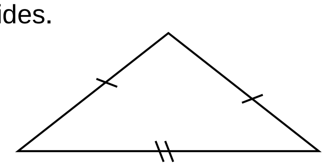
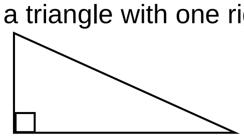
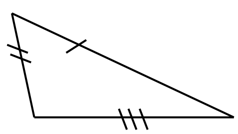
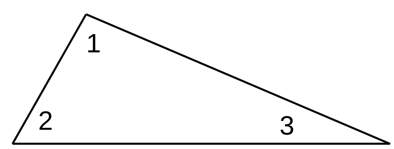
Angle Pairs	Defining Properties	Examples
complementary angles	sum of degree measures is 90°	∠ *KHF* and ∠ *KFH* (∠ 1 and ∠ 2)
supplementary angles	sum of degree measures is 180°	∠ *ACB* and ∠ *BCE* (∠ 4 and ∠ 6)
adjacent angles	two angles that share a common vertex and ray, and lie on opposite sides of the ray	∠ *GFK* and ∠ *KFH* (∠ 3 and ∠ 2)
vertical angles	opposite angles formed when two lines intersect	∠ *ACD* and ∠ *BCE* (∠ 5 and ∠ 6)

Some facts about angles:

Any two right angles are supplementary. This is because a right angle measures 90°, so any two right angles have measures with a sum of 180°.

In a right triangle, the two lesser angles are always complementary. This is because the sum of the measures of the angles of a triangle is 180°. Since the right angle measures 90°, the sum of the other two angles must be 90°.

<table>
<tr><th colspan="2">Classifying Triangles</th></tr>
<tr><td colspan="2">A triangle is a three-sided polygon. Triangles may be classified by their sides or by their angles.</td></tr>
<tr><th>Classification by Sides</th><th>Classification by Angles</th></tr>
<tr><td>An <u>equilateral</u> triangle is a triangle with three congruent sides.
</td><td>An <u>acute</u> triangle is a triangle with three acute angles.


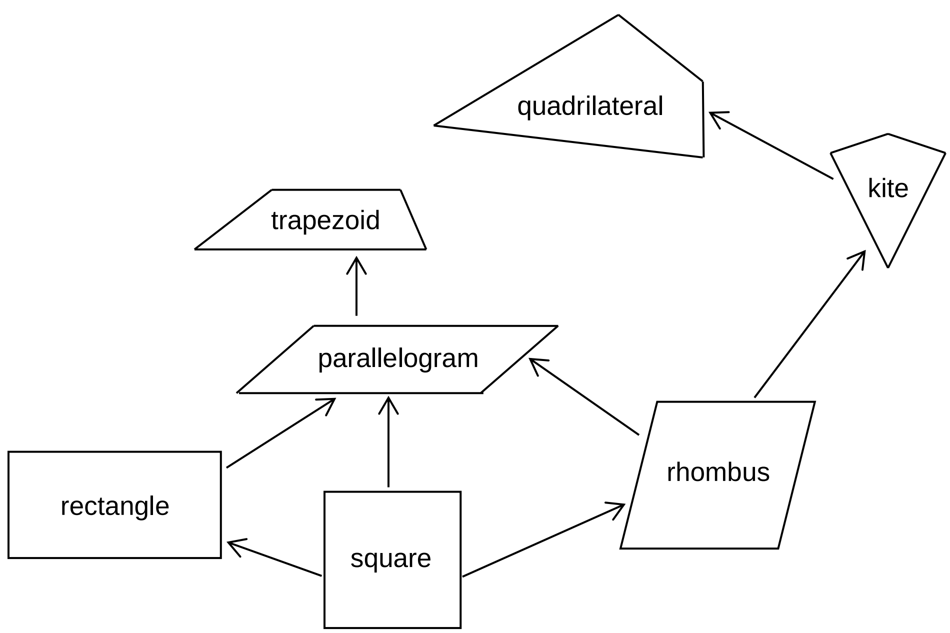
$|\angle 1| < 90^\circ$ $|\angle 2| < 90^\circ$ $|\angle 3| < 90^\circ$</td></tr>
<tr><td>An <u>isosceles</u> triangle is a triangle with at least two congruent sides.
</td><td>A <u>right</u> triangle is a triangle with one right angle.

The square in the corner indicates that the angle measures 90°.</td></tr>
<tr><td>A <u>scalene</u> triangle is a triangle with no congruent sides.
</td><td>An <u>obtuse</u> triangle is a triangle with one obtuse angle.


$|\angle 1| > 90^\circ$ $|\angle 2| < 90^\circ$ $|\angle 3| < 90^\circ$</td></tr>
<tr><td colspan="2">Note that an equilateral triangle is also <u>equiangular</u> because all three angles measure 60°.</td></tr>
</table>

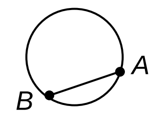
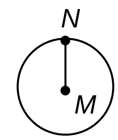
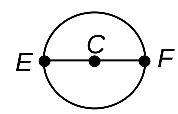
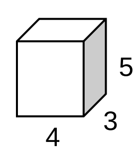
Some Properties of Quadrilaterals

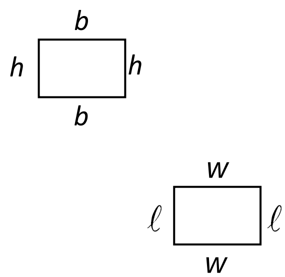
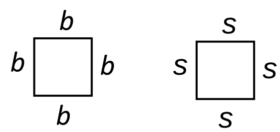
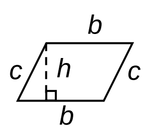
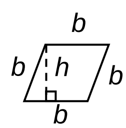
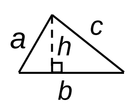
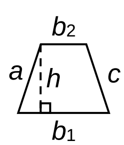
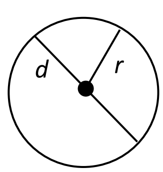
A quadrilateral is a four-sided polygon. Some of the common types of quadrilaterals are:

rectangle	A quadrilateral with four right angles. Opposite sides of a rectangle are parallel and have the same length.
square	A quadrilateral with four congruent sides and four right angles. A square is a rectangle.
parallelogram	A quadrilateral in which opposite sides are parallel. Opposite sides of a parallelogram have the same length, and opposite angles have the same measure.
rhombus	A quadrilateral whose four sides have the same length. A square is a rhombus, but a rhombus is not necessarily a square. (The plural of "rhombus" is either "rhombuses" or "rhombi.")
trapezoid	A quadrilateral with at least one pair of parallel sides.
kite	A quadrilateral whose four sides can be grouped in two pairs of adjacent sides of the same length. The two vertices where the congruent sides meet determine a line of symmetry of the kite.



STUDENT RESOURCES

Word or Phrase	Definition
center of a circle	See circle.
chord	A chord of a circle is a line segment whose endpoints lie on the circle. If the chord passes through the center of the circle, it is a diameter of the circle. The segment from A to B is a chord. 
circle	A circle is a closed curve in a plane consisting of all points at a fixed distance (the radius) from a specified point (the center). The center is at M and the radius is the length of the line segment from M to N. 
circumference	The circumference of a circle is the length of the circle, that is, the distance around it. The circumference of a circle of radius r is $C = 2\pi r$. See circle.
diameter	A diameter of a circle is a line segment joining two points of the circle that passes through the center of the circle. The line segment from E to F is a diameter. 
pi	Pi (written π) is the Greek letter used to denote the value of the ratio of the circumference of a circle to its diameter. Pi is an irrational number, with decimal representation $\pi = 3.14159\ldots$. The rational numbers 3.14 and $\frac{22}{7}$ are often used to approximate π .
radius	A radius of a circle is a line segment from the center of the circle to a point on the circle. The radius of a circle also refers to the length of that line segment. See circle.
surface area	The surface area of a three-dimensional figure is a measure of the size of the surface of the figure, expressed in square units. If the surface of the three-dimensional figure consists of two-dimensional polygons, the surface area is the sum of the areas of the polygons. A rectangular box has a length of 3”, width of 4”, and height of 5”. Surface Area = 2(3 • 4) + 2(3 • 5) + 2(4 • 5) = 94 square inches 
volume	The volume of a three-dimensional figure is a measure of the size of the figure, expressed in cubic units. A rectangular box has a length of 3”, width of 4”, and height of 5”. Volume = (3)(4)(5) = 60 cubic inches

Summary of Perimeter and Area Formulas			
Shape/Definition	**Diagram**	**Perimeter or Circumference**	**Area**
Rectangle a quadrilateral with 4 right angles		$P = 2b + 2h$ or $P = 2\ell + 2w$	$A = bh$ or $A = \ell w$
Square a rectangle with 4 equal side lengths		$P = 4b$ or $P = 4s$	$A = b^2$ or $A = s^2$
Parallelogram a quadrilateral with opposite sides parallel		$P = 2(b + c)$ or $P = 2b + 2c$	$A = bh$
Rhombus a quadrilateral with 4 equal side lengths		$P = 4b$	$A = bh$
Triangle a polygon with three sides		$P = a + b + c$	$A = \frac{1}{2}bh$
Trapezoid a quadrilateral with at least one pair of parallel sides		$P = a + b_1 + b_2 + c$	$A = \frac{1}{2}(b_1 + b_2)h$
Circle a closed figure in a plane where all points are a fixed distance (radius) from a given point (center)		$C = 2\pi r$ or $C = \pi d$	$A = \pi r^2$
For consistency, we illustrate all formulas using *b* to refer to the length of a base. The consistent use of *b* makes the relationships among formulas more apparent.			

About Pi

Pi (also written as the Greek letter π) is the value of the ratio of the circumference of a circle to its diameter. The constant π is slightly greater than 3, so that the circumference of a circle is a little more than 3 times its diameter.

Though we often use 3.14 or $\frac{22}{7}$ for the value of π , these are only approximations. It can be shown that π is not a rational number. That is, pi cannot be represented as a quotient of two integers. The decimal expansion of pi is nonrepeating (no repeating pattern exists).

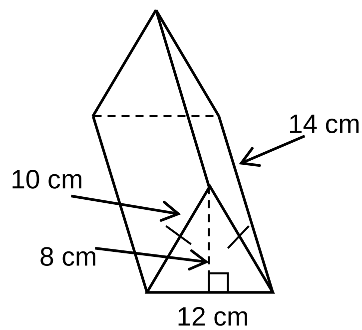
$$\pi = 3.14159265358979323846264338327950288419 71\ldots$$

Right Prisms

Every right prism has two faces (the Bases) that are congruent parallel polygons, and lateral faces that are rectangles.

Pictured below is a right triangular prism. It has two congruent parallel triangular bases and three faces that are rectangles. It is sitting on one of its lateral faces.

The height of the prism is the distance from one base to the other.



Problem: Find its surface area and volume.

Solution Step 1: Define variables.

Let b = length of triangular base = 12 cm

Let h = height of triangular base = 8 cm

Let H = height of right prism = 14 cm

Step 2: Find the volume.

To find the volume (V) of any right prism, multiply the area of the base (B) by the height (H) of the prism.

$V = BH$

$V = \left(\frac{1}{2}bh\right)H$

$V = \left(\frac{1}{2}\bullet 12\bullet 8\right)\bullet 14$

$V = (48)(14) = 672\text{ cm}^3$

Step 3: Find the surface area.

To find the surface area (SA) of any right prism, add the areas of the faces.

Find the area of the triangular base (two of these):

$A = \frac{1}{2}bh$

$A = \frac{1}{2}\bullet 12 \bullet 8 = 48\text{ cm}^2$

Find the area of the 10 cm × 14 cm rectangular face (two of these):

$A = \ell w = 10 \bullet 14 = 140\text{ cm}^2$

Find the area of the 12 cm × 14 cm rectangular face (one of these):

$A = \ell w = 12 \bullet 14 = 168\text{ cm}^2$

Finally add areas of the faces.

$SA = 48 + 48 + 140 + 140 + 168$

$SA = 544\text{ cm}^2$

STUDENT RESOURCES

Word or Phrase	Definition
experimental probability	In a repeated probability experiment, the experimental probability of an event is the number of times the event occurs divided by the number of trials. This is also called empirical probability. If, in 25 rolls of a number cube, we obtain an even number 11 times, we say that the experimental probability of rolling an even number is $\frac{11}{25}$ = 0.44 = 44%.
measure of center	A measure of center is a statistic describing the middle of a numerical data set. The mean, the median, and the mode are three commonly used measures of center. For the data set {3, 3, 5, 6, 6}, the mean (average) is $\frac{(3+3+5+6+6)}{5}$ = 4.6, and the median is 5. There are two modes, 3 and 6. Each of these numbers can be viewed as the "center" of the data set in some way.
measure of spread	A measure of spread (or a measure of variability) is a statistic describing the variability of a numerical data set. It describes how far the values in a data set are from the mean or median. The standard deviation (SD or σ), the mean absolute deviation (MAD), and the interquartile range (IQR) are three measures of spread.
population	The population is the entire group of individuals (objects or people) to which a statistical question refers. If a survey is taken to investigate how many pets the students at Seaside School own, the population under study is the entire student body of Seaside School.
sample	A sample is a subset of the population that is examined in order to make inferences about the entire population. The sample size is the number of elements in the sample. In order to estimate how many phones coming off the production line were defective, the plant manager randomly selected a sample of 50 phones and tested them to see if they worked properly.
simulation	Simulation is the imitation of one process by means of another process. We may simulate rolling a number cube by drawing a card blindfold from a group of six identical cards labeled one through six. We may simulate the weather by means of computer models.
theoretical probability	The theoretical probability of an event is a measure of the likelihood of the event occurring. In the probability experiment of rolling a (fair) number cube, there are six equally likely outcomes, each with probability $\frac{1}{6}$. Since the event of rolling an even number corresponds to 3 of the outcomes, the theoretical probability of rolling an even number is 3 out of 6, or $3 \bullet \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

Dot Plots (Line Plots)

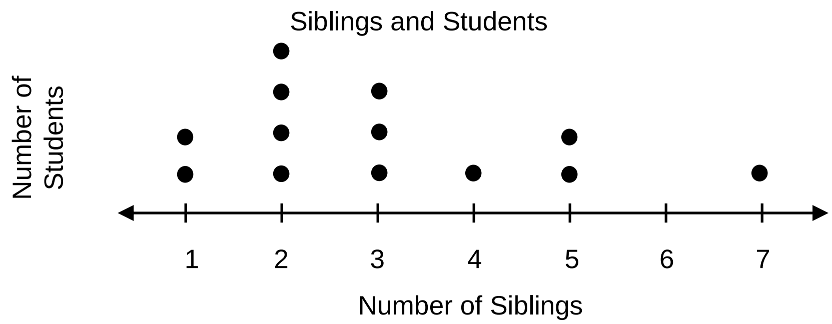
A dot plot (also called a line plot) displays data on a number line with a dot (•) or an X to show the frequency of data values.

Here are the number of siblings (brothers and sisters) for 13 different students:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To make a dot plot of this data set:

- Make a number line that extends from the minimum data value to the maximum data value
- Mark a dot or an X for every data value
- Write a title and add vertical and horizontal labels



Measures of Center

Here are the number of siblings for 13 different students:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To find the mean (average) of a data set, add all the values in the data set and divide the total by the number of values (number of observations, n).

Number of observations: $n = 13$
To find the mean: $3 + 4 + 5 + 2 + 2 + 3 + 3 + 2 + 2 + 5 + 7 + 1 + 1 = 40$
$40 \div 13 \approx 3.08$

To find the median (M), order the values from least to greatest and find the middle number. If there is an even number of values in the data set, the median is the mean (average) of the two middle numbers.

For the siblings data set: {1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 5, 5, 7}

↑ median (3)

To find the mode, find the value that occurs most often. (Some data sets may have more than one mode.)

For the siblings data set, the mode is 2. It is the value 2 occurs most often.

The Range, the Quartiles, and the Five-Number Summary

Here are the number of siblings for 13 different students:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To find the range of a data set, find the difference between the greatest value and the least value in the data set.

For the siblings data set, the range is 6, since 7 – 1 = 6.

To find quartiles, first put the numbers in numerical order. Then locate the points that divide the data set into four equal parts.

1	1	2	2	2	2	3	3	3	4	5	5	7

↑ *min* ↑ Q_1 ↑ $Q_2 = M$ median ↑ Q_3 ↑ *max*

For the siblings data set: $Q_1 = 2$ (the 1st quartile)
$Q_2 = 3$ (the 2nd quartile; also the median)
$Q_3 = 4.5$ (the 3rd quartile)

Q_1 is the median of the first half of the data set, and Q_3 is the median of the second half.

The <u>five-number summary</u> is (min, Q_1, Q_2, Q_3, max) = (1 , 2 , 3 , 4.5 , 7).

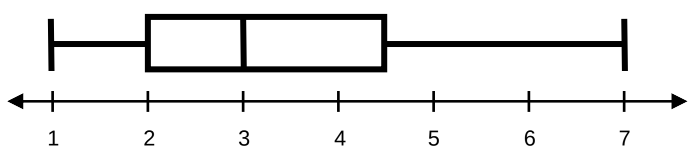
Box Plots (Box-and-Whisker Plots)

A box plot (or box-and-whisker plot) provides a visual representation of the center and spread of a data set. The display is based on the five-number summary.

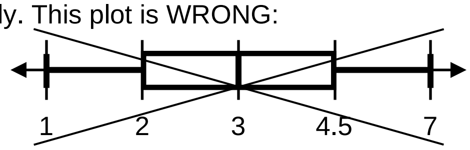
For the sibling data, the five-number summary is (1 , 2 , 3 , 4.5 , 7).

To make a box plot:

- Locate the five-number summary values on a number line, and indicate each value with a vertical segment
- Create a "box" to highlight the interval from the first to the third quartile, and draw "whiskers" that extend to the minimum and maximum

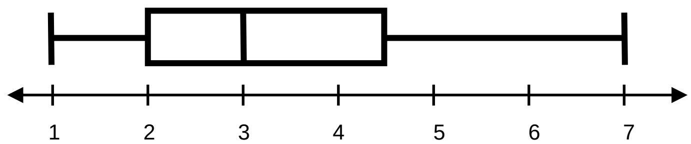


Be sure to scale the box plot properly. This plot is WRONG:



Interpreting a Box Plot

This is a box plot.



- Each of the four "sections" (the two whiskers and the two rectangular parts of the box) contains (close to) one-fourth of the data points. Be careful: If one section appears larger than another, we cannot say it has more data points, but only that the data points are spread out over a wider range.

- Sometimes we use the word "quartile" to refer to specific data points. Sometimes the word "quartile" is also used to refer to one of the four quarters, or sections, of the data set. For example, data points that lie within the farthest left section may be referred to as "in the first quartile."

Mean Absolute Deviation

The mean absolute deviation (MAD) is a measure of spread of a numerical data set. It is the arithmetic average of the distance (absolute value) of each data point to the mean. To calculate the MAD statistic:

For the sibling data, there are 13 data points:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To find the MAD statistic:

- Find the mean of the sample.

  The mean is 3.08.

- Find the distance (absolute value) from each data point to the mean.

  See the table entries to the right.

- Find the sum of the distances.

  See the bottom row of the table.

- Divide the sum of the distances by the number of data points to find the average distance from the mean.

  See the calculation below.

Sibling Data	**Distance from data point to mean**
3	$\lvert 3.08 - 3 \rvert = 0.08$
4	$\lvert 3.08 - 4 \rvert = 0.92$
5	$\lvert 3.08 - 5 \rvert = 1.92$
2	$\lvert 3.08 - 2 \rvert = 1.08$
2	$\lvert 3.08 - 2 \rvert = 1.08$
3	$\lvert 3.08 - 3 \rvert = 0.08$
3	$\lvert 3.08 - 3 \rvert = 0.08$
2	$\lvert 3.08 - 2 \rvert = 1.08$
2	$\lvert 3.08 - 2 \rvert = 1.08$
5	$\lvert 3.08 - 5 \rvert = 1.92$
7	$\lvert 3.08 - 7 \rvert = 3.92$
1	$\lvert 3.08 - 1 \rvert = 2.08$
1	$\lvert 3.08 - 1 \rvert = 2.08$
Sum of distances from mean	17.4

$$\text{MAD} = \frac{\text{sum of distances from mean}}{\text{number of data points}} = \frac{17.4}{13} = 1.34$$

Sampling

Sampling refers to selecting a subset of a population to be examined for the purpose of drawing statistical inferences about the entire population. If the sample is representative of the entire population, we may make valid inferences about the entire population based on properties of the sample.

Suppose you want to know how many hours per week students in our school spend watching television. From the population of all students, you select a sample and you ask the students in the sample how many hours they watch television. You would like to infer that the average time spent watching TV for all students is about the same as for students in the sample.

An easy way to select a sample might be to ask your friends how many hours they watch TV. Such a sample is called a convenience sample. However, your friends may not be representative of all students.

To select a more representative sample, you might assign a number to the name of each student in the school, and then use a computerized "random number generator" to select a certain number of the students' numbers. This sort of sample is referred to as a random sample. Its mathematical properties allow us to draw inferences about the population.

The Percent Error Formula

A percent error computation is used to quantify the difference between an estimated value and an actual value as a percentage of the actual value. The formula for percent error (as a percent) is:

$$\text{Percent Error}_{(\text{as a percent})} = \frac{|\text{actual} - \text{estimate}|}{\text{actual}}\text{, written as a percent}$$

If the estimated number of fish in a lake is 45, and the actual number is 50, then the percent error is:

$$\frac{|50 - 45|}{50} = \frac{|5|}{50} = \frac{1}{10} = 10\%$$

If the estimated number of fish in a lake is 50, and the actual number is 45, then the percent error is:

$$\frac{|45 - 50|}{50} = \frac{|-5|}{50} = \frac{1}{10} = 10\%$$