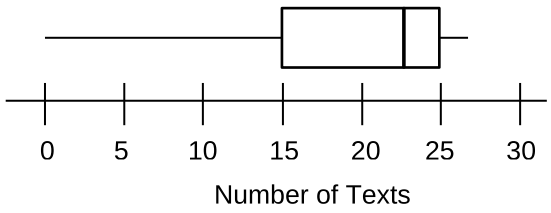
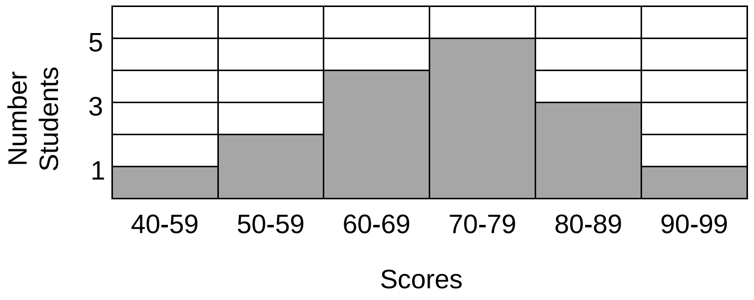
TEACHER EDITION

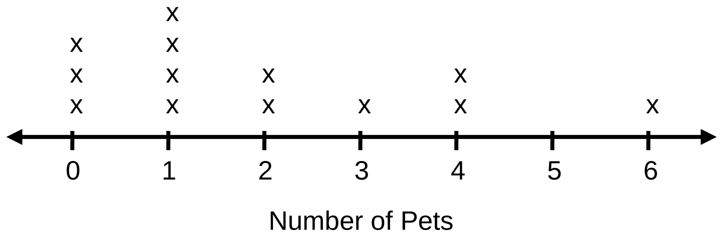
MathLinks GRADE 6

COMPILED STUDENT RESOURCES

UNIT 1	1
UNIT 2	8
UNIT 3	15
UNIT 4	21
UNIT 5	29
UNIT 6	35
UNIT 7	40
UNIT 8	43
UNIT 9	48
UNIT 10	53

STUDENT RESOURCES

Word or Phrase	Definition
box plot	A <u>box plot</u>, or <u>box-and-whiskers plot</u>, is a graphical representation of the five-number summary of a data set. See <u>five-number summary</u>. Box Plot of Number of Texts Per Day of 6th Graders 
dot plot	A <u>dot plot</u> is a graphical representation of a data set where the data values are represented by dots above a number line. See <u>line plot</u>.
five-number summary	The <u>five-number summary</u> of a data set consists of its minimum value (min), first quartile Q_1, median Q_2, third quartile Q_3, and maximum value (max). The five-number summary is usually written in the form (min, Q_1, med., Q_3, max). The five-number summary of the data set 1, 1, 1, 3, 5, 5, 6, 7, 23 is given by (min, Q_1, med., Q_3, max) = (1, 1, 5, 6.5, 23).
histogram	A <u>histogram</u> is a graphical representation of frequencies of a numerical variable using rectangles. For a histogram, the horizontal axis is divided into intervals. Each interval forms the base of a rectangle whose height corresponds to the frequency of values of the variable in that interval. Quiz Scores of a Class of 16 Students 
interquartile range	The <u>interquartile range</u> (IQR) of a numerical data set is the difference between the third quartile and the first quartile of the data set. The interquartile range is a measure of the variation of the data set. For the data set 1, 1, 1, 3, 5, 5, 6, 7, 23, $Q_1 = 1$, $Q_3 = 6.5$, and IQR = 5.5

Word or Phrase	Definition
line plot	A <u>line plot</u> is a graphical representation of a data set where the data values are represented by marks, such as dots or X's, above a number line. See <u>dot plot</u>. Line Plot of Number of Pets for 13 Students 
mean	The <u>mean</u> of a data set is a measure of center equal to the average of the values in the data set. The mean is calculated by adding the values in the data set and dividing by the number of data values. The mean of the data set 1, 1, 1, 3, 5, 5, 6, 7, 23 is $\frac{1+1+1+3+5+5+6+7+23}{9} = 5\frac{7}{9} = 5.77...$
mean absolute deviation	The <u>mean absolute deviation</u> (MAD) of a data set is the average of the (positive) differences between the values in the data set from the mean. The MAD is a measure of the variation of the data set. For the data set {3, 3, 5, 6, 6}, the mean is 4.6. The distances of the data points to the mean are 1.6, 1.6, 0.4, 1.4, and 1.4. The MAD is $\frac{1.6 + 1.6 + 0.4 + 1.4 + 1.4}{5} = 1.28$
measure of center	A <u>measure of center</u> is a statistic describing the middle of a data set. The mean, the median, and the mode are three commonly used measures of center of a numerical data set.
measure of spread	A <u>measure of spread</u> is a statistic describing the variability of a data set. It describes how far the values in a data set are from the mean or median. The standard deviation, the mean absolute deviation (MAD), and the interquartile range (IQR) are three measures of spread of a numerical data set.
median	The <u>median</u> of a data set is a measure of center equal to the middle number in the data set, when the values are placed in order from least to greatest. If there is an even number of values in the data set, the median is taken to be the mean (average) of the two middle values. The median of the data set 1, 1, 1, 3, 5, 5, 6, 7, 23 is 5, since the first 5 is the middle value. The median of the data set 5, 6, 7, 23 is the mean (average) of the two middle numbers, (6 + 7) ÷ 2 = 6.5, which is the average of 6 and 7.

Word or Phrase	Definition
mode	The mode of a data set is the value(s) that occur(s) most often. A data set may have more than one mode. It may also have no mode if all values occur the same number of times. The mode of the data set 1, 1, 1, 3, 5, 6, 6, 7, 23 is 1, since the data value 1 occurs more frequently than any other data value. If a 6 were added to this data set, 6 would also be a mode.
outlier	An outlier of a data set is a data value that is a striking deviation from the overall pattern of values in the data set. For the data set 1, 1, 1, 3, 5, 6, 6, 7, 23, the data value 23 is a potential outlier. It appears unusually large relative to the other data values.
quartiles	The quartiles of a data set are points that divide the data set into four equally sized groups, when the values are placed in order from least to greatest. The second quartile is the median, denoted by Q_2. The first quartile, denoted by Q_1, is the median of the lower half of the data set (the data values less than the middle data value), and the third quartile, denoted by Q_3, is the median of the upper half of the data set. Given the ordered data set 1, 1, 1, 3, 5, 5, 6, 7, 23, • The middle value is the first 5: Median = 5. This is also the second quartile Q_2, • The lower half of the data set is 1, 1, 1, 3. Therefore $Q_1 = 1$. • The upper half of the data set is 5, 6, 7, 23. Therefore, $Q_3 = 6.5$.
range (of a data set)	The range of a numerical data set is the difference between the greatest and least values in the data set. The range of the data set 1, 1, 1, 3, 5, 5, 6, 7, 23 is 22, since 22 = 23 – 1.
statistical question	A statistical question is a question where numerical data that has potential for variability can be collected and analyzed for the purpose of answering the question. A statistical question: “How much TV do students in my class watch on average?” NOT a statistical question: “How many hours of TV did you watch last week?”

Finding Measures of Center

Here are the number of siblings for 13 different students:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To find the median, order the value from least to greatest and find the middle number. If there is an even number of values in the data set, the median is the mean (average) of the two middle numbers.

The median for the siblings data set: $1, 1, 2, 2, 2, 2, \boxed{3,}\ 3, 3, 4, 5, 5, 7$

To find the mode, find the value(s) that occur(s) most often.

The mode for the siblings data set: the value of 2 occurs most often.

To find the mean (average) of a data set, add all the values in the data set and divide it by the number of values (number of observations, *n*).

Number of observations: $n = 13$

The mean for the siblings data set: $\dfrac{3+4+5+2+2+3+3+2+2+5+7+1+1}{13} = 3.08$

Finding the Range and the Quartiles

Here are the number of siblings for 13 different students:

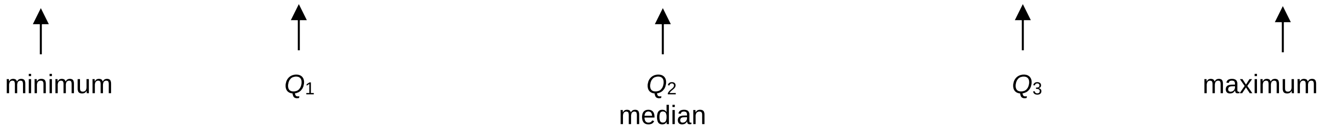
3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

To find the range of a data set, find the difference between the greatest and least values in the data set.

For the siblings data set, the range is 6, since 7 – 1 = 6

To find quartiles, first put the numbers in numerical order. Then locate the points that divide the set into four equal parts.

1	1	2	2	2	2	3	3	3	4	5	5	7



For the siblings data set:

$Q_1 = 2$ (the 1st quartile)

$Q_2 = 3$ (the 2nd quartile)

$Q_3 = 4.5$ (the 3rd quartile)

Note that Q_1 is the median of the first half of the data set and Q_3 is the median of the second half.

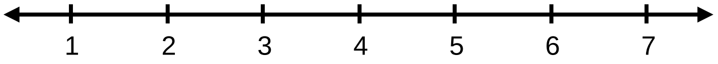
How to Construct a Dot Plot

A dot plot (also called a line plot) displays data on a number line with a dot (•) or an X to show the frequency of data values.

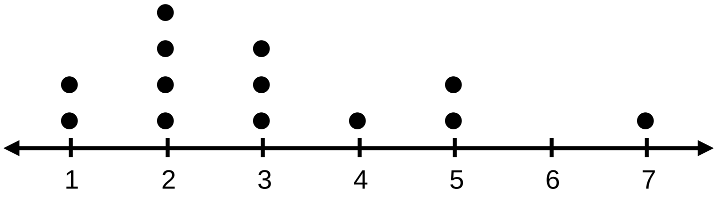
Here are the number of siblings for 13 different students:

3, 4, 5, 2, 2, 3, 3, 2, 2, 5, 7, 1, 1

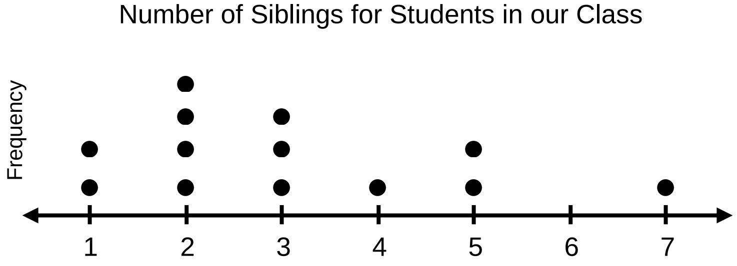
1. Make a number line that extends from the minimum data value to the maximum data value.



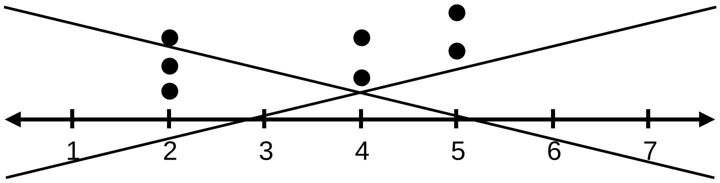
2. Mark a dot or an X for every data value.



3. Write a title and labels.



Heads Up! Be sure to line up dots or X's properly. The graph below is visually misleading in a few places. The number of dots at 4 and 5 are the same, but one set is higher than the other, possibly implying there are more. The number of dots at 2 and 4 are different, but they peak at the same height, possibly implying there are the same number of dots.



How to Construct a Histogram

A histogram is a data display that uses adjacent rectangles to show the frequency of data values in intervals. The height of a given rectangle shows the frequency of data values in the interval shown at the base of the rectangle.

Nancy asks each of her 21 classmates how many coins they have in their backpacks. Then she puts the data set in order.

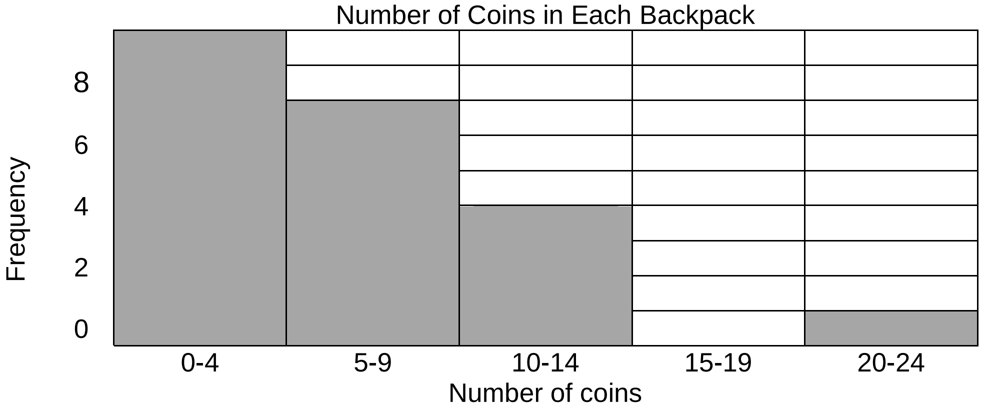
0, 0, 1, 2, 2, 2, 2, 3, 3, 5, 5, 7, 7, 7, 7, 7, 10, 10, 10, 12, 21

To construct the histogram:

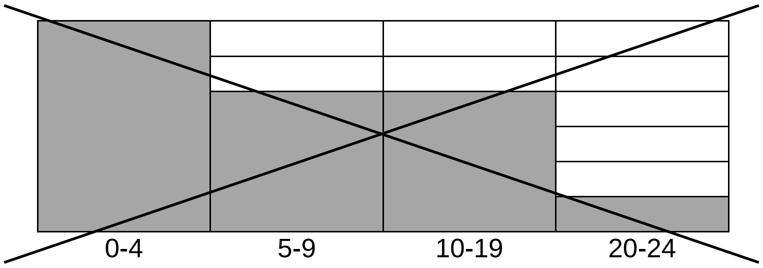
1. Divide the number of coins into equally spaced intervals and make a frequency table: (Here we choose intervals of five.)

Intervals (number of coins)	Frequency
0-4	9
5-9	7
10-14	4
15-19	0
20-24	1

2. Record frequencies as rectangles on a data display. Add a title and label the axes.



Heads Up! Be sure to make equally spaced intervals. The graph below is visually misleading. The third column has an interval that is twice the others, but the same number of data points as the column to the left of it.



How to Construct a Box Plot

A box plot (or box-and-whisker plot) is a visual representation of the center and spread of a data set. The display is based on the five-number summary.

Here are the ages of 15 people:

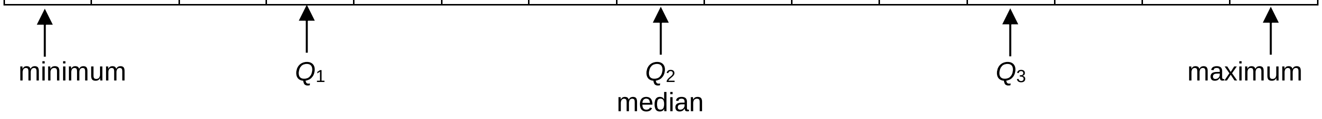
21, 12, 28, 17, 46, 35, 7, 38, 42, 33, 19, 9, 31, 25, 28

1. Write the values of the data set from least to greatest.

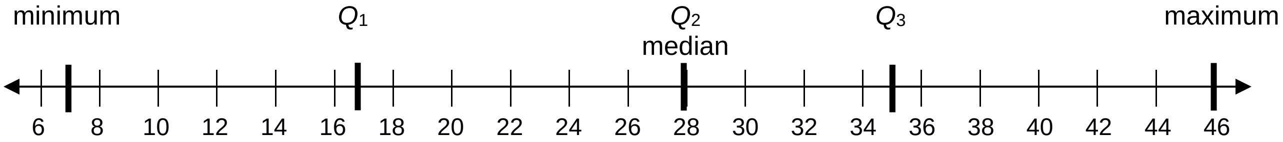
7, 9, 12, 17, 19, 21, 25, 28, 28, 31, 33, 35, 38, 42, 46

2. Find the five-number summary.

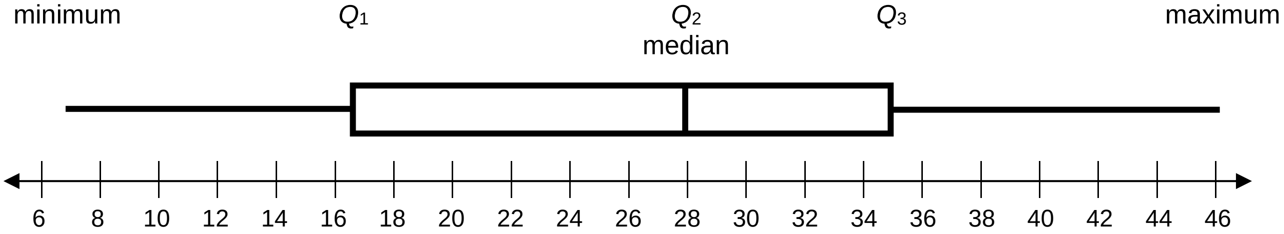
7	9	12	17	19	21	25	28	28	31	33	35	38	42	46



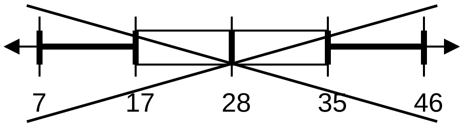
3. Locate the five-number summary values on a number line, and indicate with vertical segments.



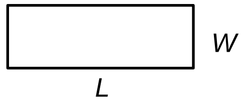
4. Create a "box" to highlight the interval from the first to the third quartile, and draw "whiskers" that extend to the minimum and maximum.

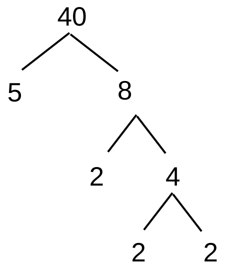
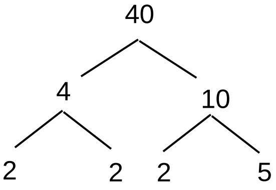


Heads Up! Be sure to scale the box and whisker plot properly. This plot is WRONG:



STUDENT RESOURCES

Word or Phrase	Definition
area	The area of a two-dimensional figure is a measure of the size of the figure, expressed in square units. The area of a rectangle is the product of its length and its width. If a rectangle has a length of 12 inches and a width of 5 inches, its area is (5)(12) = 60 square inches. Area = Length × Width = $L \bullet W$ 
composite number	A number is composite if it has more than two divisors or factors. 12 has six factors: 1, 2, 3, 4, 6, 12, because 12 = 1 • 12, 12 = 2 • 6, and 12 = 3 • 4. Since 12 has more than two factors, 12 is a composite number.
factor (of a number)	A factor of a number is a divisor of the number. The factors of 12 are 1, 2, 3, 4, 6, and 12.
greatest common factor	The greatest common factor (GCF) of two numbers is the greatest factor that divides the two numbers. The factors of 12 are 1, 2, 3, 4, 6, and 12. The factors of 18 are 1, 2, 3, 6, 9, and 18. Therefore, the GCF of 12 and 18 is 6.
least common multiple	The least common multiple (LCM) of two numbers is the least number that is a multiple of both numbers. The multiples of 8 are 8, 16, 24, 32, 40, ... The multiples of 12 are 12, 24, 36, 48, … . Therefore, the LCM of 8 and 12 is 24.
lowest common denominator	The lowest common denominator of two fractions is the least common multiple of their denominators. The lowest common denominator of $\frac{3}{8}$ and $\frac{5}{12}$ is 24.
multiple (of a number)	A multiple of a number m is a number of the form $k \bullet m$ for any integer k. The numbers 5, 10, 15, and 20 are multiples of 5, since 1 • 5 = 5, 2 • 5 = 10, 3 • 5 = 15, and 4 • 5 = 20.
natural number	The natural numbers are the numbers 1, 2, 3, 4, ….

Word or Phrase	Definition
prime factorization	The <u>prime factorization</u> of a number is an expression of that number as a product of primes. There is a unique way to express any number as a product of primes, except for order.   40 = 5 • 2 • 2 • 2 and 40 = 2 • 2 • 2 • 5 The two <u>prime factorization trees</u> above illustrate that even though the order of the prime factors is different, the products are the same.
prime number	A <u>prime number</u> is a natural number that has exactly two factors, namely 1 and itself. The first six prime numbers are 2, 3, 5, 7, 11, and 13. 1 is *not* a prime number. It has exactly one factor.
relatively prime	Two numbers are <u>relatively prime</u> if their greatest common factor is 1. The factors of 6 are 1, 2, 3, and 6. The factors of 11 are 1 and 11. Since the greatest common factor of 6 and 11 is 1, the two numbers are relatively prime.
square number	A <u>square number</u>, or <u>perfect square</u>, is a number that is a square of a natural number. The area of a square with natural number side-length is a square number. The square numbers are $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, … .

Symbols for Multiplication

The product of 8 and 4 can be written as:

8 times 4

8 × 4

8 • 4

(8)(4)

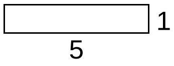
$$\begin{array}{r} 8 \\ \times\ \ 4 \\ \hline \end{array}$$

In algebra, we generally avoid using the × for multiplication because it could be misinterpreted as the variable *x*, and we cautiously use the symbol • for multiplication because it could be misinterpreted as a decimal point.

Using Rectangles to Visualize Prime and Composite Numbers

Building rectangles whose sides have natural number lengths is a geometric way to describe factors and multiples of numbers. If the area of the rectangle represents the product, then the side lengths of the rectangle represent the factors of the number.

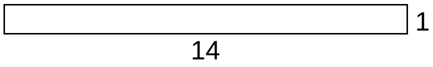
A <u>prime number</u> p corresponds to only one rectangle, since p can be factored as a product in only one way, $p = 1 \bullet p$. (Here we regard the factorization $p = 1 \bullet p$ as the same as $p = p \bullet 1$, and we regard a $1 \times p$ rectangle as being the same as a $p \times 1$ rectangle.)

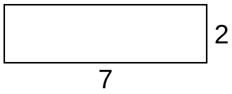


5 = 1 × 5

1 and 5 are factors of 5.

A <u>composite number</u> n always corresponds to more than one rectangle.



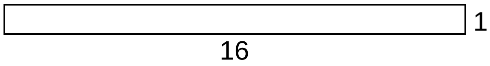


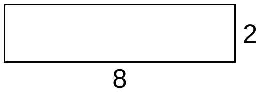
14 = 1 × 14

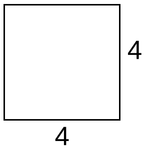
14 = 2 × 7

1, 2, 7, and 14 factors of 14.

A number such as 16 is called a <u>square number</u> (or <u>perfect square</u>) because one of the rectangles it corresponds to is a square (4 × 4).







16 = 1 × 16

16 = 2 × 8

16 = 4 × 4

1, 2, 4, 8, and 16 factors of 16.

Greatest Common Factor (GCF)

The greatest common factor (GCF) of two numbers is the greatest factor that divides the two numbers. Here are two ways to find the GCF of two numbers.

Tensaye has 12 bottles of water and 18 granola bars. She wants to use all of the bars and bottles to make care packages for the homeless. How many care packages can Tensaye make so that there are the same number of bottles of water and granola bars in each package?

Method 1: Use a list to find the GCF of 12 and 18:

List all the factors of 12: 1, 2, 3, 4, 6, and 12
List all the factors of 18: 1, 2, 3, 6, 9, and 18

We can see that the factors 1, 2, 3, and 6 appear in both lists. Since 6 is the greatest factor from both lists that divides 12 and 18, the greatest common factor (GCF) of 12 and 18 is 6.

Method 2: Use prime factorization to identify the GCF of 12 and 18.

$$\begin{array}{r} 12 = \\ 18 = \end{array} \boxed{\begin{array}{c} 2 \\ 2 \end{array}} \begin{array}{c} \times\ 2\ \times \\ \times\ 3\ \times \end{array} \boxed{\begin{array}{c} 3 \\ 3 \end{array}}$$

We see that 2 and 3 are common factors of both numbers. Therefore $2 \times 3 = 6$ is the GCF.

Since the GCF of 12 and 18 is 6, Tensaye can make 6 care packages for the homeless, and each care package will contain 2 bottles of water and 3 granola bars.

Least Common Multiple (LCM)

The least common multiple (LCM) of two numbers is the least number that is a positive multiple of both numbers. Here is one way to find the LCM of two numbers.

Tensaye wants buy bottles of water and granola bars to make care packages for the homeless. Bottles of water come in packages of 12, and granola bars are sold in packages of 18. How many bottles of water and how many granola bars should Tensaye buy so that she has the same number of each item? Note; She can only afford to buy the smallest amount to make this happen.

Use a list to find the LCM of 12 and 18:

The multiples of 12 are: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120,… .

The multiples of 18 are: 18, 36, 54, 72, 90, 108, 126, 144, 162, 180, ...

The multiples that 12 and 18 have in common are 36, 72, 108, ... We can see that 36 is the least multiple the two numbers have in common. Therefore, the LCM of 12 and 18 is 36.

Since the LCM of 12 and 18 is 36, Tensaye should buy 36 bottles (or 3 packages) of water and 36 granola bars (or 2 packages) so that she has the same number of each item.

The "Big One"

The "Big 1" is a notation for 1 (multiplicative identity) in the form of a fraction $\frac{n}{n}$ ($n \neq 0$).

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \dots$$

We can use the following picture to help remind us that these fractions are equivalent to 1:

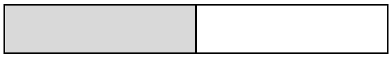
$$1 = \boxed{\frac{8}{8}}$$

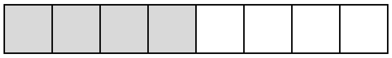
The Big 1 can be used to help find equivalent fractions. For example,

$$\frac{2}{5} \times \boxed{\frac{10}{10}} = \frac{20}{50} \qquad \text{or} \qquad \frac{20}{50} \div \boxed{\frac{10}{10}} = \frac{2}{5}.$$

Diagrams that Show Equivalent Fractions

These diagrams illustrate that $\frac{1}{2} = \frac{4}{8}$. In the second diagram, each half is split into four parts, but the size of the whole does not change, nor does the amount shaded.





Using the Big 1, this equivalence can be written:

$$\frac{1}{2} \bullet \boxed{\frac{4}{4}} = \frac{4}{8} \qquad \text{or} \qquad \frac{4}{8} \div \boxed{\frac{4}{4}} = \frac{1}{2}$$

Fractions in "Simplest Form" with the GCF

To write a fraction in its simplest form, divide the numerator and denominator by the greatest common factor. Though it is not required to use the GCF, doing so is the most efficient way, because it only takes one step. Use the Big 1 when dividing.

To simplify $\frac{12}{30}$, first use any method to determine that the GCF of 12 and 30 is 6. Then divide the numerator and denominator by 6, in the form of the Big 1.

$$\frac{12}{30} \div \boxed{\frac{6}{6}} = \frac{2}{5}$$

Renaming Fractions with their Lowest Common Denominator (LCD)

To rename fractions with their LCD, first find the least common multiple (LCM) of the denominators. Then change each fraction to an equivalent fraction by multiplying each of them by the appropriate forms of the Big 1.

To write two fractions, $\frac{3}{4}$ and $\frac{5}{6}$, with their LCD, first find the LCM of the denominators. After using any method to determine that the LCD of 4 and 6 is 12, rename the fractions so that they both have a denominator of 12 using the Big 1.

$$\frac{3}{4} \times \frac{3}{3} = \frac{9}{12} \qquad \frac{5}{6} \times \frac{2}{2} = \frac{10}{12}$$

This computation results in lesser numerators and denominators in the fractions because 12 is the least mutiple that 4 and 6 share in common. In other words, 12 is the LCM of 4 and 6, or the LCD of the fractions.

Using “Factor Ladders” to Find the GCF and LCM of Two Numbers

Factor ladders are a useful tool for finding the GCF and LCM of two numbers.

Use repeated division to find the GCF and LCM of 12 and 18:

Divide each number by any common factor greater than 1. In this case we have choices, so let’s begin by dividing both numbers by 2. The resulting quotients are 6 and 9.

Keep dividing until both resulting quotients have no factors in common greater than 1. In this case, we can still divide by 3. The resulting quotients are now 2 and 3, and they have no common factors greater than 1. They are relatively prime.

$$\begin{array}{r|cc} 2 & 12 & 18 \\ \hline 3 & 6 & 9 \\ \hline & 2 & 3 \end{array}$$

The GCF is the product of the factors along the side. Therefore, the GCF of 12 and 18 is $2 \bullet 3 = 6$.

The LCM is the product of the factors along the side and the bottom. Therefore, the LCM of 12 and 18 is the GCF multiplied by 2 and 3, or $6 \bullet 2 \bullet 3 = 36$.

Fraction Addition with Diagrams

The standard procedure for adding fractions requires that the fractions have common denominators. An area model supports why this is reasonable.

Example 1: $\frac{1}{5} + \frac{3}{5}$

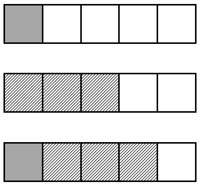
The fractions already have a common denominator.

$\frac{1}{5} + \frac{3}{5} = \frac{4}{5}$

$\frac{1}{5}$

$\frac{3}{5}$

$\frac{1}{5} + \frac{3}{5}$



Example 2: $\frac{1}{2} + \frac{1}{3}$

Find a common denominator.

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{2} \bullet \boxed{\frac{3}{3}} + \frac{1}{3} \bullet \boxed{\frac{2}{2}}$$

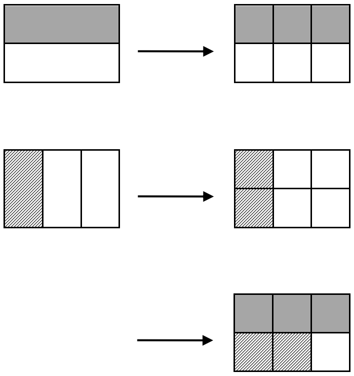
$$= \frac{3}{6} + \frac{2}{6}$$

$$= \frac{5}{6}$$

$\frac{1}{2} = \frac{3}{6}$

$\frac{1}{3} = \frac{2}{6}$

$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$



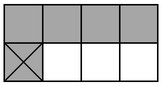
Fraction Subtraction with Diagrams

The standard procedure for subtracting fractions requires that the fractions have common denominators. An area model supports why this is reasonable.

Example 1: $\frac{5}{8} - \frac{1}{8}$

The fractions already have a common denominator.

$\frac{5}{8} - \frac{1}{8} = \frac{4}{8}$.



Example 2: $\frac{1}{2} - \frac{1}{3}$

Find a common denominator.

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{2} \bullet \boxed{\frac{3}{3}} - \frac{1}{3} \bullet \boxed{\frac{2}{2}}$$

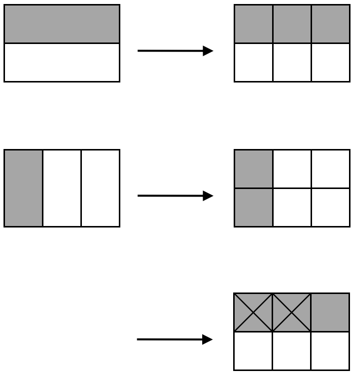
$$= \frac{3}{6} - \frac{2}{6}$$

$$= \frac{1}{6}$$

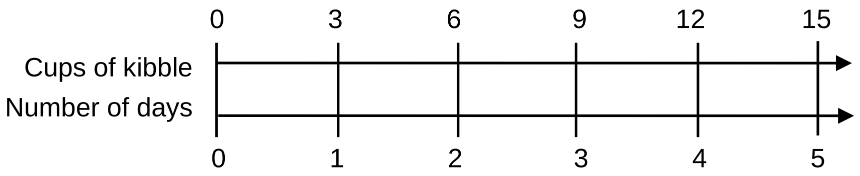
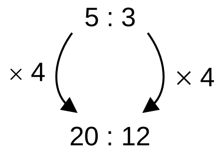
$\frac{1}{2} = \frac{3}{6}$

$\frac{1}{3} = \frac{2}{6}$

$\frac{3}{6} - \frac{2}{6} = \frac{1}{6}$



STUDENT RESOURCES

Word or Phrase	Definition
conversion rate	A conversion rate is a unit rate expressing the number of units of one measure equal to one unit of another. Two conversion rates are 1.3 dollars per euro and 60 minutes per hour.
customary units	In the United States, customary units are a system of units of measurement that includes ounces, pounds, and tons to measure weight; inches, feet, yards, and miles to measure length; pints, quarts, and gallons to measure capacity; and degrees Fahrenheit to measure temperature.
double number line	A double number line is a diagram made up of two parallel number lines that visually depict the relative sizes of two quantities. Double number lines are often used when the two quantities have different units, such as miles and hours. The proportional relationship “Wrigley eats 3 cups of kibble per day” can be represented in the following double number line diagram. 
equivalent ratios	Two ratios are equivalent if each number in one ratio is obtained by multiplying the corresponding numbers in the other ratio by the same positive number. 5 : 3 and 20 : 12 are equivalent ratios because both numbers in the ratio 5 : 3 are multiplied by 4 to get to the ratio 20 : 12. An arrow diagram can be used to show equivalent ratios. 
metric units	Metric units are a system of units of measurement that includes grams and kilograms to measure weight; millimeters, centimeters, meters, and kilometers to measure length; milliliters and liters to measure capacity; and degrees Celsius to measure temperature.
ratio	A ratio is a pair of positive numbers in a specific order. The ratio of *a* to *b* is denoted by *a* : *b* (read “*a* to *b*,” or “*a* for every *b*”). The ratio of 3 to 2 is denoted by 3 : 2. The ratio of dogs to cats is 3 to 2. There are 3 cups of water for every 2 cups of juice. The fraction $\frac{3}{2}$ does not represent this ratio, but it does represent the *value of the ratio* (or the unit rate).

Word or Phrase	Definition
tape diagram	A tape diagram is a graphical representation that uses length to represent relationships between quantities. We draw rectangles with a common width to represent quantities, and rectangles with the same length to represent equal quantities. Tape diagrams are typically used to represent quantities expressed in the same unit. This tape diagram represents a drink mixture with 3 parts grape juice for every 2 parts water. 
unit price	A unit price is a price for one unit of measure. If 4 apples cost \$1.00, then the unit price is $\frac{\$1.00}{4}$ = \$0.25 for one apple, or 0.25 dollars per apple or 25 cents per apple.
unit rate	The unit rate associated with a ratio a : b of two quantities *a* and *b*, $b \neq 0$, is the number $\frac{a}{b}$, to which units may be attached. This is sometimes referred to as the value of the ratio. The ratio of 40 miles for every 5 hours has a unit rate of $\frac{40}{5} = 8$ miles per hour.
value of a ratio	See unit rate.

Ratios: Language and Notation

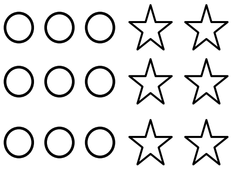
The ratio of *a* to *b* is denoted by $a : b$ (read " *a* to *b*," or "*a* for every *b*").

Note that the ratio of *a* to *b* is not the same as the ratio of *b* to *a* unless $a = b$.

We can identify several ratios for the objects in the picture to the right.

- There are 3 circles for every 2 stars.
- The ratio of stars to circles is 2 to 3.
- The ratio of circles to total shapes is 3 : 5
- The ratio of total shapes to stars is 5 : 2.

Three copies of the figure above are pictured to the right. Here the ratio of circles to stars is 9 : 6. The ratio 9 : 6 is obtained by multiplying each number in the ratio 3 : 2 by 3 (called the multiplier).



circles : stars

$$\begin{tikzcd} 3 : 2 \arrow[d, bend right, "\times 3"'] \arrow[d, bend left, "\times 3"] \\ 9 : 6 \end{tikzcd}$$

The arrow diagram to the left shows that 3 : 2 and 9 : 6 are equivalent ratios.

A fraction formed by a ratio is called the value of the ratio (or unit rate). Equivalent ratios have the same value. In our example, $\frac{3}{2} = \frac{9}{6} = 1.5$.

Tables of Number Pairs

Tables are useful for recording number pairs that have equivalent ratios. In the case of a ratio of three circles for every two stars, there are two ways that number pairs with equivalent ratios might be recorded in a table. Table 1 is aligned horizontally. Table 2 is aligned vertically. Entries may be in any order.

○○○☆☆

Table 1

Circles	3	9	6
Stars	2	6	4

Table 2

Circles	**Stars**
3	2
9	6
6	4

Tape Diagrams

A <u>tape diagram</u> is a visual model consisting of strips divided into rectangular segments whose areas represent relative sizes of quantities. Tape diagrams are typically used when quantities have the same units.

This tape diagram shows that the ratio of grape juice to water in some mixture is 2 : 4.

G	*G*	*W*	*W*	*W*	*W*

Suppose we want to know how much grape juice is needed to make a mixture that is 24 gallons. Here are two methods:

Method 1:

G	*G*	*W*	*W*	*W*	*W*
G	*G*	*W*	*W*	*W*	*W*
G	*G*	*W*	*W*	*W*	*W*
G	*G*	*W*	*W*	*W*	*W*

Replicate the tape diagram, making 24 rectangles. Each rectangle now represents 1 gallon. This shows that:

2 gallons grape : 4 gallons water (6 total gallons)

is the same ratio as

8 gallons grape : 16 gallons water (24 total gallons)

24 gallons of mixture will require 8 gallons of grape juice.

Notice here that each rectangle (piece of tape) represents 1 unit (1 gallon of liquid.)

Method 2:

G	*G*	*W*	*W*	*W*	*W*

<table>
<tr><td colspan="6">24 gallons</td></tr>
<tr><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td><td>4</td></tr>
<tr><td colspan="2">8 gal</td><td colspan="4">16 gal</td></tr>
</table>

Six rectangles in the tape diagram represent 24 gallons of mixture.

Since 24 ÷ 6 = 4, one rectangle in the tape diagram represents 4 gallons of liquid.

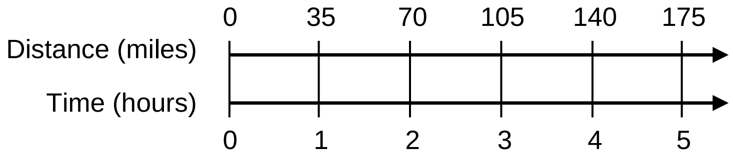
24 gallons of mixture will require 8 gallons of grape juice.

Notice here that each rectangle (piece of tape) represents more than 1 unit (4 gallons, in this case). Pieces of tape in the diagrams do not always need to represent 1 unit.

Double Number Lines

A double number line diagram is a graphical representation of two quantities in which corresponding values are placed on two parallel number lines for easy comparison. Double number lines are often used to compare two quantities that have different units.

The double number line below shows corresponding ratios if a car travels 70 miles every 2 hours.



We can see from the double number line diagram above that at the given rate, the car goes 35 miles in 1 hour (which is the unit rate of 35 miles per hour), 105 miles in 3 hours, etc. Notice the same tick marks on the number line are used to represent different quantities, and values are scaled in numerical order.

Unit Rate and Unit Price

The unit rate associated with a ratio is the value of the ratio, to which we usually attach units for clarity. In other words, the unit rate associated with the ratio $a : b$ is the number $\frac{a}{b}$, to which we may attach units. For this to make sense, we must assume that $b \neq 0$.

Suppose a car travels 70 miles every 2 hours.

- This may be represented by the ratio 70 : 2.
- The number $\frac{70}{2} = \frac{35}{1} = 35$ is the value of the ratio.
- The unit rate is then the value 35, to which we attach the units “miles per hour.” Thus, the unit rate may be written:

$35 \frac{\text{miles}}{\text{hour}}$ or 35 miles per hour or 35 miles/hour

A unit price is the price for one unit.

Suppose it costs $1.50 for 5 apples.

- This may be represented as the ratio 1.50 : 5.
- The number $\frac{1.50}{5} = 0.30$ is the value of the ratio.
- The unit price is then the value 0.30, to which we attach the units “dollars per apple.” The unit price can be written in any of the forms below.

$0.30 \frac{\text{dollars}}{\text{apple}}$ 0.30 dollars per apple $0.30 per apple

Metric Measurements	
Common metric units	**Examples (sizes approximate)**
Length	
1 millimeter (mm)	the thickness of a dime
1 centimeter (cm)	the width of a small finger
1 meter (m)	the length of a baseball bat
1 kilometer (km)	the length of 9 football fields
Capacity / Volume	
1 milliliter (mL)	an eyedropper
1 liter (L)	a juice carton
1 kiloliter (kL)	four filled bathtubs
Mass / Weight	
1 milligram (mg)	a grain of sand
1 gram (g)	a paperclip
1 kilogram (kg)	a textbook

U.S. Customary Measurements	
Common customary units	**Examples (sizes approximate)**
Length	
1 inch (in)	the length of a small paperclip
1 foot (ft)	the length of a sheet of notebook paper
1 yard (yd)	the width of a door
1 mile (mi)	the length of 15 football fields
Capacity / Volume	
1 fluid ounce (fl oz)	a serving of honey
1 cup (c)	a small cup of coffee
1 pint (pt)	a bowl of soup
1 quart (qt)	an engine oil container
1 gallon (gal)	a jug of milk
Mass / Weight	
1 ounce (oz)	a slice of bread
1 pound (lb)	a soccer ball
1 bushel (bsh)	a block of hay
1 ton (T)	a walrus

Conversion Statements

Length

1 foot = 12 inches
1 yard = 3 feet
1 mile = 5,280 feet
1 kilometer = 1,000 meters
1 meter = 100 centimeter

1 centimeter ≈ 0.4 inches
1 meter ≈ 39 inches
1 kilometer ≈ 0.6 mile

Area

1 acre = 43,560 square feet

Capacity / Volume

1 cup = 8 fluid ounces
1 pint = 2 cups
1 quart = 4 cups
1 gallon = 4 quarts

1 liter ≈ 1.06 quarts

Mass / Weight

1 pound = 16 ounces
1 bushel = 60 pounds
1 ton = 2,000 pounds
1 kilogram = 1,000 grams
1 kilogram ≈ 2.2 pounds

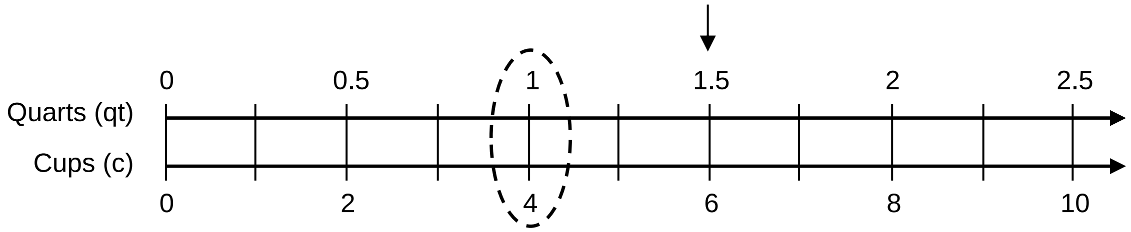
Conversions

Double number lines can be used to organize measurement conversion calculations.

How many cups are in 1.5 quarts?

Create a double number line that shows 1 quart = 4 cups.

Then fill in other numbers on the line to answer the question.



There are 6 cups in 1.5 quarts.

Information from a double number line may also be organized into a table.

STUDENT RESOURCES

Word or Phrase	Definition
conjecture	A conjecture is a statement that is proposed to be true, but has neither been proven to be true nor to be false.
dividend	In a division problem, the dividend is the number being divided. In $12 \div 3 = 4$, the dividend is 12. dividend ÷ divisor = quotient
divisor	In a division problem, the divisor is the number by which another is divided. In $12 \div 3 = 4$, the divisor is 3. $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient}$
multiplication property of 1	The multiplication property of 1 states that $a \bullet 1 = 1 \bullet a = a$ for all numbers a. In other words, 1 is a multiplicative identity. The multiplicative property of 1 is sometimes called the multiplicative identity property. $4 \bullet 1 = 4$, $1 \bullet (\frac{3}{8}) = \frac{3}{8}$, $\frac{3}{4} \bullet \frac{5}{5} = \frac{15}{20} = \frac{3}{4}$
quotient	In a division problem, the quotient is the result of the division. In $12 \div 3 = 4$, the quotient is 4. $\begin{array}{r}\text{quotient}\\ \text{divisor}\overline{)\text{dividend}}\end{array}$
reciprocal	For $b \neq 0$, the reciprocal of b is the number, denoted by $\frac{1}{b}$, that satisfies $b \bullet \frac{1}{b} = 1$. The reciprocal of b is also called the multiplicative inverse of b. The reciprocal of 3 is $\frac{1}{3}$. The reciprocal of $\frac{1}{6}$ is 6. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.
unit rate	The unit rate associated to a ratio $a : b$, where a and b have units attached, is the number $\frac{a}{b}$, with the units "a-units per b-unit" attached. The ratio of 400 miles for every 8 hours corresponds to the unit rate 50 miles per hour.

Notation for Division

The quotient of 8 and 4 can be written as:

8 divided by 4 $8 \div 4$ $4\overline{)8}$ $\frac{8}{4}$ 8/4

In algebra, the preferred way to show division is with fraction notation.

A Chunking Division Procedure

This chunking division procedure keeps the dividend intact as we “close in” on the quotient. If you do not know all your multiplication facts, this procedure may be easier than the standard division algorithm because you subtract out groups of the divisor more flexibly, but still arrive at the correct quotient. If the largest amount possible is chosen to subtract at each step, this procedure is very efficient.

Divide 761 highlighters into 3 boxes.

Step 1: Rewrite problem

$$\begin{array}{r|l} 3\overline{)761} & \\ & \\ & \end{array}$$

Step 2: Make a Multiplication Bank that may be useful for this problem.

3 × 1 = 3	3 × 10 = 30	3 × 100 = 300
3 × 2 = 6	3 × 20 = 60	3 × 200 = 600
3 × 3 = 9	3 × 30 = 90	3 × 300 = 900
3 × 4 = 12	3 × 40 = 120	3 × 400 = 1200

Step 3: Select a fact from the Multiplication Bank that is less than or equal to the dividend, and record. Continue the routine until the remainder is less than the divisor.

$$\begin{array}{r|l} 3\overline{)\;761} & \\ -\underline{600} & 200 \\ 161 & \end{array}$$

$$\begin{array}{r|l} 3\overline{)\;761} & \\ -\underline{600} & 200 \\ 161 & \\ -\underline{120} & 40 \\ 41 & \\ & \end{array}$$

$$\begin{array}{r|l} 3\overline{)\;761} & \\ -\underline{600} & 200 \\ 161 & \\ -\underline{120} & 40 \\ 41 & \\ -\underline{30} & 10 \\ 11 & \\ & \end{array}$$

$$\begin{array}{r|r} 253 & \text{R } 2 \\ 3\overline{)\;761} & \\ -\underline{600} & 200 \\ 161 & \\ -\underline{120} & 40 \\ 41 & \\ -\underline{30} & 10 \\ 11 & \\ -\underline{9} & \underline{+3} \\ 2 & 253 \end{array}$$

The last calculation shows that the quotient is (200 + 40 + 10 + 3) = 253, and the remainder is 2.

The Standard Division Algorithm for Whole Numbers

The standard division algorithm is an efficient process for dividing. It involves a cyclical process: divide, multiply, subtract, “bring down”… until the remainder is less than the divisor.

$14\overline{)\underline{96}\,3}$	Determine where to start	Look at the divisor. Choose digits in the dividend so that the quotient using these digits is between 1 and 9.
$\begin{array}{r} 6 \\ 14\overline{)963} \end{array}$	Divide	How many 14s in 96? Write this number above the 96. Place value reminder: The 96 in the dividend represents 960. The 6 in the quotient represents 60.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ \underline{-84} \end{array}$	Multiply	Find the product of 6 and 14. Write this below the 96. Place value reminder: 6 × 14 = 84 is compact notation for 60 ×14 = 840.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ \underline{-84} \\ 12 \end{array}$	Subtract	Find the difference between 96 and 84. Write this below the 84. Place value reminder: 96 – 80 = 12 is compact notation for 960 – 840 = 120.
$\begin{array}{r} 6 \\ 14\overline{)963} \\ \underline{-84}\downarrow \\ 123 \end{array}$	Bring down	Bring down the next digit.
$\begin{array}{r} 68 \\ 14\overline{)963} \\ \underline{-84}\downarrow \\ 123 \\ \underline{-112} \\ 11 \end{array}$	Divide Multiply Subtract Bring down (remainder)	Repeat the divide, multiply, subtract, bring down (if necessary) process until the remainder is less than the divisor.

Some ways to represent the dividend, divisor, quotient, and remainder:

$\begin{array}{rl} \text{quotient} & \text{remainder} \\ \text{divisor}\overline{)\text{dividend}} & \end{array}$

dividend = (divisor)(quotient) + remainder

$\begin{array}{rl} 68 & \text{R11} \\ 14\overline{)963} & \end{array}$

$\begin{array}{rl} 68 & \frac{11}{14} \\ 14\overline{)963} & \end{array}$

963 = (14)(68) + 11

Why Do We Move the Decimal Point when Dividing Decimals?

The procedure for dividing decimals involves “moving the decimal point.” The reason this is done is because we usually consider dividing by a whole number to be an easier process.

Consider $12.5 \div 0.25$, which can be written as $0.25\overline{)12.5}$ or $\frac{12.5}{0.25}$.

Since $12.5 \div 0.25$ may be multiplied by 1 in the form of $\boxed{\frac{100}{100}}$, it is equal to $\frac{12.5}{0.25} \bullet \frac{100}{100} = 1{,}250 \div 25$.

Now we can divide by a whole number. This process often is depicted this way:

$$0.25\overline{)12.5} \rightarrow \underset{\curvearrowright\quad\curvearrowright}{0.25\overline{)12.5\underline{0}}} \rightarrow \begin{array}{r} . \\ 025.\overline{)1250.} \end{array} \rightarrow \begin{array}{r} 50. \\ 25.\overline{)1250.} \end{array}$$

Division of Decimals: Examples

- Multiply the divisor and dividend by the same power of 10 (10, 100, 1000, etc.) so that the divisor is a whole number.

- Divide as usual, lining up the digits of the quotient above the dividend so that the tens line up with tens, ones with ones, tenths with tenths, and so on. Place the decimal in the quotient in the same location as the dividend.

To obtain more decimal place accuracy, attach zeroes to the right of the final place in the decimal part and continue dividing until the remainder is zero (example 2) or the quotient pattern repeats (example 3).

Example 1

$$\underset{\curvearrowright\qquad\quad\curvearrowright}{0.02\overline{)0.358}}$$

$$\begin{array}{r} 17.9 \\ 2\overline{)35.8} \\ \underline{-2}\downarrow \\ 15 \\ \underline{-14}\downarrow \\ 18 \\ \underline{-18} \\ 0 \end{array}$$

Example 2

$$8\overline{)3} \rightarrow 8\overline{)3.000}$$

$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{-24}\downarrow \\ 60 \\ \underline{-56}\downarrow \\ 40 \\ \underline{-40} \\ 0 \end{array}$$

Example 3

$$11\overline{)4} \rightarrow 11\overline{)4.0000}$$

$$\begin{array}{r} 0.3636... \\ 11\overline{)4.0000} \\ \underline{-33}\downarrow \\ 70 \\ \underline{-66}\downarrow \\ 40 \\ \underline{-33}\downarrow \\ 70 \\ \underline{-66} \\ 4 \end{array}$$

Standard Algorithms for Decimal Operations	
Addition - Set up the problem in columns, with place values lined up to add tens with tens, ones with ones, tenths with tenths, etc. When the digits are properly lined up, the decimal points will also align. - (Optional) Include trailing zeroes to the right of the decimal points as place holders if needed, as in this problem where 1 thousandth is added to 0 thousandths. - Add with regrouping as usual. Since the place values in the sum line up with the place values in the two addends, the decimal point in the sum will align with the decimal points in the addends.	$\begin{array}{r} \overset{1}{4}\overset{1}{8}.560 \\ +36.521 \\ \hline 85.081 \end{array}$
Subtraction - Set up the problem in columns, with place values lined up to subtract tens from tens, ones from ones, tenths from tenths, etc. When the digits are properly lined up, the decimal points will also align. - Include trailing zeroes to the right of the decimal point as place holders in the minuend (top number) as needed to line up with any trailing nonzero digit in the subtrahend (bottom number). - Subtract as though the decimal points are not there. When done calculating, place the decimal point in the difference directly below the decimal points in the problem.	$\begin{array}{r} \overset{6}{7}.\overset{13}{4}\overset{10}{0} \\ -3.51 \\ \hline 3.89 \end{array}$
Multiplication - Set up the problem in columns, with digits right justified. - Ignore decimal placement and multiply. - Place decimal in the product. The number of digits to the right of the decimal point in the product is equal to the *sum* of the number of digits to the right of the decimal point of each factor.	$\begin{array}{rl} 30.5 & \text{(1 decimal place)} \\ \times\ \underline{0.003} & \text{(3 decimal places)} \\ 0.0915 & \text{(4 decimal places)} \end{array}$
Division - Multiply the divisor and dividend by the same power of 10 (10, 100, 1000, etc.) so that the divisor is a whole number. - Divide as usual, lining up the digits of the quotient above the dividend so that the tens line up with tens, ones with ones, tenths with tenths, and so on. Place the decimal in the quotient in the same location as the dividend. To obtain more decimal place accuracy, attach zeroes to the right of the final place in the decimal part and continue dividing until the remainder is zero or the quotient pattern repeats.	$0.25\overline{)12.5} \quad \rightarrow$ $0.25\overline{)12.5\underline{0}} \quad \rightarrow$ $\begin{array}{r} . \\ 025.\overline{)1250.} \end{array} \quad \rightarrow$ $\begin{array}{r} 50. \\ 25.\overline{)1250.} \end{array}$

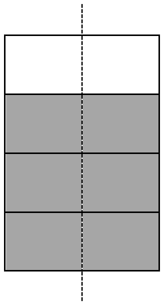
Visualizing Fraction Division as "Divvy Up"

A "divvie up" division problem poses the question:

"How can we divide ___ into ___ equal groups?

Suppose we want to divide $\frac{3}{4}$ cups of grape juice equally among two people. This division problem $\frac{3}{4} \div 2$, can be interpreted as "how can we divide $\frac{3}{4}$ into 2 equal parts?

Let the rectangle represent 1 full cup. It is filled with $\frac{3}{4}$ cups of grape juice.



From the diagram we see that each person will get $\frac{3}{8}$ cup of juice.

Therefore, $\frac{3}{4} \div 2 = \frac{3}{8}$.

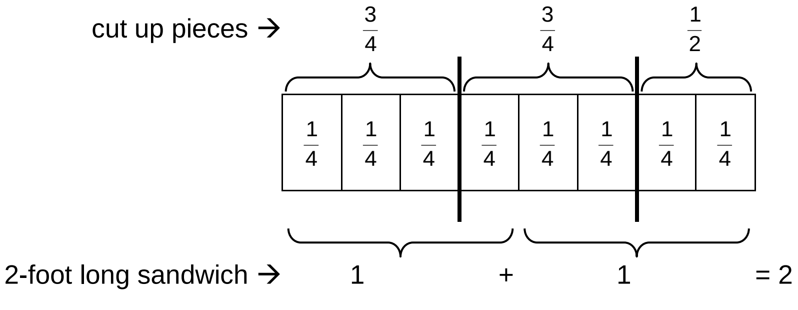
Visualizing Fraction Division as "Measure Out"

A "measure out" division problem poses the question:

"How many ___ are in ___?"

Suppose a two-foot sandwich is cut into pieces that are $\frac{3}{4}$ foot long each. This division problem $2 \div \frac{3}{4}$ can be interpreted as "how many $\frac{3}{4}$ ft. are in 2 ft.?" The unit of measure is $\frac{3}{4}$ ft. From the diagram, we see that there are TWO $\frac{3}{4}$ ft. sandwiches in the 2 ft. sandwich. We see further that there is $\frac{1}{2}$ ft. of sandwich leftover. Since $\frac{1}{2} = \frac{2}{3}$ of $\frac{3}{4}$, the leftover represents $\frac{2}{3}$ of the unit of measure.

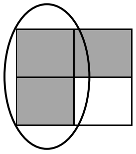
Therefore, $2 \div \frac{3}{4} = 2\frac{2}{3}$.



A Closer Look at the Unit in Fraction Measurement Division

Consider the problem: How many $\frac{1}{2}$s are in $\frac{3}{4}$?

$$\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$$



What is the whole? $\frac{3}{4}$

What is the unit of measure? $\frac{1}{2}$

Is there a full $\frac{1}{2}$ ***in*** $\frac{3}{4}$***?*** Yes.

How much is leftover? $\frac{1}{4}$

What part of the unit is leftover? $\frac{1}{2}$ because $\frac{1}{4}$ is $\frac{1}{2}$ of $\frac{1}{2}$.

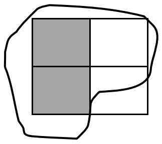
How many $\frac{1}{2}$***s are in*** $\frac{3}{4}$***?*** $1\frac{1}{2}$

$\frac{1}{2}$ is circled and $\frac{1}{4}$ is left over.
$\frac{1}{4}$ is $\frac{1}{2}$ of a $\frac{1}{2}$.

In this case, a larger positive number is being divided by a smaller positive number. The result is a quotient greater than 1.

Consider the problem: How many $\frac{3}{4}$s are in $\frac{1}{2}$?

$$\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$$



What is the whole? $\frac{1}{2}$

What is the unit of measure? $\frac{3}{4}$

Is there a full $\frac{3}{4}$ ***in*** $\frac{1}{2}$***?*** No.

How many $\frac{3}{4}$***s are in*** $\frac{1}{2}$***?*** $\frac{2}{3}$

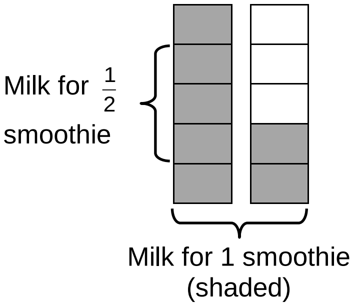
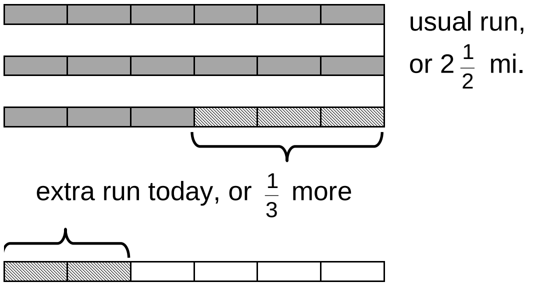
$\frac{2}{3}$ of $\frac{3}{4}$ is shaded.

In this case, a smaller positive number is being divided by a larger positive number. The result is a quotient less than 1.

Rules for Dividing Fractions

Divide Across	Multiply by the Reciprocal
$\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \bullet \frac{d}{c}$
$b \neq 0$, $d \neq 0$	$b \neq 0$, $d \neq 0$

Examples: Dividing Fractions

Words or Diagrams	Divide Across	Multiply by the Reciprocal
Millie needs $1\frac{1}{2}$ cups of milk to make a smoothie. How much smoothie can Millie make with $\frac{3}{4}$ cup of milk? 	$\frac{3}{4} \div 1\frac{1}{2}$ $= \frac{3}{4} \div \frac{3}{2}$ $= \frac{3 \div 3}{4 \div 2}$ $= \frac{1}{2}$	$\frac{3}{4} \div 1\frac{1}{2}$ $= \frac{3}{4} \div \frac{3}{2}$ `$= \frac{3}{4} \times \frac{2}{3}$ $= \frac{3 \times 2}{4 \times 3}$ $= \frac{6}{12} = \frac{1}{2}$
Helen usually runs $2\frac{1}{2}$ miles a day. Today, she ran $3\frac{1}{3}$ miles. How much of her usual run did Helen run today? 	$3\frac{1}{3} \div 2\frac{1}{2} = \frac{10}{3} \div \frac{5}{2}$ $= \frac{20}{6} \div \frac{15}{6}$ $= \frac{20 \div 15}{6 \div 6}$ $= \frac{\frac{20}{15}}{1} = \frac{20}{15}$ $= 1\frac{5}{15} = 1\frac{1}{3}$	$3\frac{1}{3} \div 2\frac{1}{2} = \frac{10}{3} \div \frac{5}{2}$ $= \frac{10}{3} \times \frac{2}{5}$ $= \frac{10 \times 2}{3 \times 5}$ $= \frac{20}{15}$ $= 1\frac{5}{15} = 1\frac{1}{3}$

STUDENT RESOURCES

Word or Phrase	Definition
equivalent fractions	The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if they represent the same point on the number line. This occurs if the results of the division problems $a \div b$ and $c \div d$ are equal. Since $\frac{1}{2} = 1 \div 2 = 0.5$ and $\frac{2}{4} = 2 \div 4 = 0.5$, the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent.
multiplication property of 1	The multiplication property of 1 states that $a \bullet 1 = 1 \bullet a = a$ for all numbers a. In other words, 1 is a multiplicative identity. The multiplication property of 1 is sometimes called the multiplicative identity property. $4 \bullet 1 = 4$, $1 \bullet 25 = 25$, $\frac{1}{2} \bullet \boxed{\frac{4}{4}} = \frac{4}{8}$ In the third equation above, since we are multiplying by 1 in the form of $\frac{4}{4}$, we refer to it as the Big 1.
percent	A percent is a number expressed in terms of the unit $1\% = \frac{1}{100} = 0.01$. Similarly, $p\% = \frac{p}{100} = p(0.01)$. One way to convert a number to a percent is to multiply the number by 1 in the form of 100%. $4 = 4 \times 100\% = 400\%$; $0.6 = 0.6 \times 100\% = 60\%$ One way to convert a percent to a number is to express $p\%$ as p hundredths. The fraction may be converted to a decimal by dividing. $15\% = \frac{15}{100} = 0.15$; $40\% = \frac{40}{100} = 0.40 = 0.4$.
percent of a number	A percent of a number is the product of the percent and the number. It represents the number of parts per 100 parts. 15% of 300 is $\frac{15}{100} \bullet 300 = 45$, or $(0.15)(300) = 45$. If 45 out of 300 students are boys, then 15 out of every 100 students are boys, and 15% of the students are boys.
ratio	A ratio is a pair of positive numbers in a specific order. The ratio of a to b is denoted by $a : b$ (read "*a* to *b*," or "*a* for every *b*").

Equivalent Fractions: The Big 1

The number 1 is called the multiplicative identity. Multiplying a fraction by any form of 1 does not change its value.

The Big 1 is a notation for 1 in the form of a fraction $\frac{n}{n}$ $(n \neq 0)$. For example,

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \ldots$$

We can use the following picture to help remind us that these fractions are equivalent to 1:

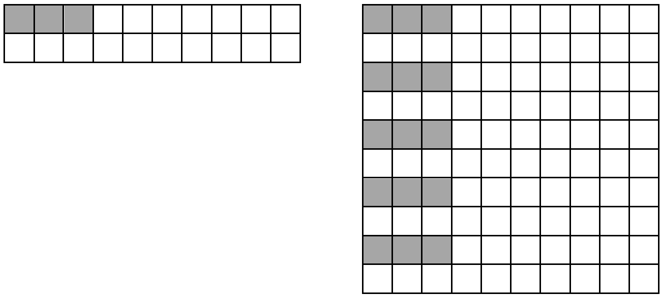
$$1 = \boxed{\frac{8}{8}}$$

The Big 1 can be used to show equivalence of fractions. For example,

$$\frac{2}{5} \times \boxed{\frac{10}{10}} = \frac{20}{50} \quad \text{or} \quad \frac{20}{50} \div \boxed{\frac{10}{10}} = \frac{2}{5}.$$

Equivalent Fractions

The diagrams below illustrate that $\frac{3}{20} = \frac{15}{100}$. In the second diagram, the pattern is repeated five times. The fractional part remains the same as the size of the whole changes.



$$\frac{3}{20} = \frac{15}{100}$$

Using the Big 1, this equivalence can be written:

$$\frac{3}{20} \bullet \boxed{\frac{5}{5}} = \frac{15}{100}.$$

Visually, multiplying the numerator by 5 represents repeating the shaded parts five times, and multiplying the denominator by 5 represents repeating the total number of parts in the denominator five times.

With this process, the size of the part does not change.

Some Fraction-Decimal-Percent Equivalents

$\frac{1}{2} = \frac{50}{100} = 0.5 = 50\%$ $\frac{1}{4} = \frac{25}{100} = 0.25 = 25\%$ $\frac{3}{4} = \frac{75}{100} = 0.75 = 75\%$ $\frac{5}{4} = \frac{125}{100} = 125 = 125\%$ Conversion strategy: Think: $\frac{3}{4}\left(\frac{25}{25}\right) = \frac{75}{100} = 75\%$	$\frac{1}{10} = \frac{10}{100} = 0.1 = 10\%$ $\frac{3}{10} = \frac{30}{100} = 0.3 = 30\%$ $\frac{5}{10} = \frac{50}{100} = 0.5 = 50\%$ Conversion strategy: Think: $\frac{3}{10} = \frac{30}{100}$, so $0.3 = 0.30 = 30\%$	$\frac{1}{25} = \frac{4}{100} = 0.04 = 4\%$ $\frac{16}{25} = \frac{64}{100} = 0.64 = 64\%$ $\frac{9}{50} = \frac{18}{100} = 0.18 = 18\%$ Conversion strategy: Think: $25(4) = 100$, so $\frac{16}{25}\left(\frac{4}{4}\right) = \frac{64}{100} = 64\%$
$\frac{3}{20} = \frac{15}{100} = 0.15 = 15\%$ $\frac{13}{20} = \frac{65}{100} = 0.65 = 65\%$ $\frac{19}{20} = \frac{95}{100} = 0.95 = 95\%$ Conversion strategy: Think: 20 nickels in a dollar $\frac{1}{20}$ of a dollar is \$0.05	$\frac{1}{5} = \frac{2}{10} = 0.2 = 20\%$ $\frac{2}{5} = \frac{4}{10} = 0.4 = 40\%$ $\frac{3}{5} = \frac{6}{10} = 0.6 = 60\%$ $\frac{4}{5} = \frac{8}{10} = 0.8 = 80\%$ Conversion strategy: Think: If I know tenths, I can easily convert to hundredths.	$\frac{1}{8} = \frac{12.5}{100} = 0.125 = 12.5\%$ $\frac{3}{8} = \frac{37.5}{100} = 0.375 = 37.5\%$ $\frac{5}{8} = \frac{62.5}{100} = 0.625 = 62.5\%$ $\frac{7}{8} = \frac{87.5}{100} = 0.875 = 87.5\%$ Conversion strategy: Think: $\frac{1}{4} = \frac{25}{100}$, so half of $\frac{1}{4}$ is $\frac{1}{8} = \frac{12.5}{100}$ $= 12.5\%$

Connecting Multiplication and Division to Percent of a Number

Think	**Example**
Finding 100% of something is the same as finding all of it.	100% of \$80 = \$80 $\underbrace{\boxed{\quad 100\% \quad}}_{\$80}$
Finding 50% of something is the same as finding one-half of it. This is the same as multiplying by $\frac{1}{2}$ or dividing by 2.	50% of \$80 = $\frac{1}{2}$(\$80) = \$40 \$80 ÷ 2 = \$40 $\underbrace{\boxed{\quad 50\% \quad}\boxed{\quad 50\% \quad}}_{\$80}$
Finding 25% of something is the same as finding one-fourth of it. This is the same as multiplying by $\frac{1}{4}$ or dividing by 4.	25% of \$80 = $\frac{1}{4}$(\$80) = \$20 \$80 ÷ 4 = \$20 $\underbrace{\boxed{25\%}\boxed{25\%}\boxed{25\%}\boxed{25\%}}_{\$80}$
Finding 10% of something is the same as finding one-tenth of it. This is the same as multiplying by $\frac{1}{10}$ or dividing by 10.	10% of \$80 = $\frac{1}{10}$(\$80) = \$8 \$80 ÷ 10 = \$8
Finding 1% of something is the same as finding one-hundredth of it. This is the same as multiplying by $\frac{1}{100}$ or dividing by 100.	1% of \$80 = $\frac{1}{100}$(\$80) = \$0.80 \$80 ÷ 100 = \$0.80
Finding 20% of something is the same as doubling 10% of it.	20% of \$80 = 2(\$8) = \$16
Finding 5% of something is the same halving 10% of it.	5% of \$80 = $\frac{1}{2}$(\$8) = \$4
Finding 15% of something is the same as adding 10% of it and 5% of it.	15% of \$80 = \$8 + \$4 = \$12

Using Chunking to Find a Percent of a Number

We use the word "chunking" to describe a process of decomposing and composing numbers to make calculations easier, especially when done mentally. Another way to describe this is "taking numbers apart and putting them back together." For example, if adding 17 and 26, we might decompose each number into tens and ones, adding 10 + 20 = 30, and 7 + 6 = 13, and finalizing the sum by adding 30 + 13 = 43.

Longer method (applying the distributive property)	**Shorter method** (mostly done mentally)
20% of 60 = (10% + 10%) of 60 = 10% of 60 + 10% of 60 = 6 + 6 = 12	20% of 60 10% → 6 10% → 6 20% → 12 Note that given a number representing the whole, we use an arrow to efficiently show the percent of the number using mental math and chunking.
15% of 60 = (10% + 5%) of 60 = 10% of 60 + 5% of 60 = 6 + 3 = 9	15% of 60 10% → 6 5% → 3 15% → 9

Using Multiplication to Find a Percent of a Number

Some percent values are hard to find mentally. For example, finding 17% of something is the same as finding $\frac{17}{100}$ = 0.17 of it. In this case, it may be easier to find the percent by using the definition of a percent of a number:

A <u>percent of a number</u> is the product of the percent and the number.

Find 17% of $80.

Strategy 1: **Use fractions**

$$\frac{17}{100}\bullet 80 = \frac{17\bullet 80}{100} = \frac{1360}{100} = 13.60$$

So 17% of \$80 is \$13.60.

Strategy 2: **Use decimals**

$(0.17) \bullet (80) = 13.6$ or 13.60

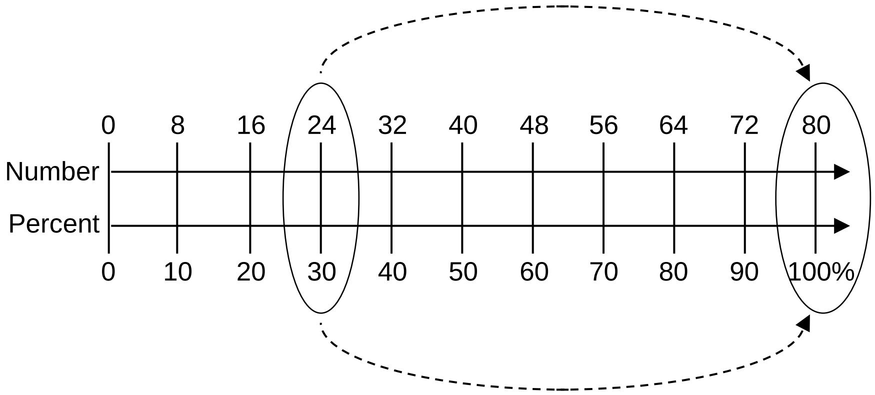
So 17% of \$80 is \$13.60.

Using Double Number Lines to Solve Percent Problems

Strategy 1: **Solve on the double number line**

30% of 80 is what amount?

Create a double number line with percents represented in increments of 10% on the bottom line, and the whole number represented in equal increments on the top. Since the whole is 80 (in this case), count by 8s for the equal increments (80 ÷ 10 = 8).



Since 30% corresponds to 24 on the double number line, 30% of 80 is 24.

Strategy 2: **Identify equivalent ratios on the double number line and create equivalent fractions.**

Create equivalent fractions based on the part-to-whole relationships.

$$\frac{\text{part}_{number}}{\text{whole}_{number}} = \frac{\text{part}_{percent}}{\text{whole}_{percent}}$$

$$\frac{\boxed{24}}{80} = \frac{30}{100}$$

This equivalence is based on the dotted arrows above.

Create equivalent fractions based on the part-to-part relationships.

$$\frac{\text{part}_{number}}{\text{part}_{percent}} = \frac{\text{whole}_{number}}{\text{whole}_{percent}}$$

$$\frac{\boxed{24}}{30} = \frac{80}{100}$$

This equivalence is based on the circles above.

STUDENT RESOURCES

Word or Phrase	Definition
coefficient	A coefficient is a number or constant factor in a term of an algebraic expression. In the expression $3x + 5$, 3 is the coefficient of the linear term $3x$, and 5 is the constant coefficient.
constant term	A constant term in an algebraic expression is a term that has a fixed numerical value. In the expression $5 + 2x + 3$, the terms 5 and 3 are constant terms. If this expression is rewritten as $2x + 8$, the term 8 is the constant term of the new expression.
distributive property	The distributive property states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c. $3(4 + 5) = 3(4) + 3(5)$; $(4 + 5)8 = 4(8) + 5(8)$; $6(8 – 1) = 6(8) – 6(1)$
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
equivalent expressions	Two mathematical expressions are equivalent if, for any possible substitution of values for the variables, the two resulting numbers are equal. In particular, two numerical expressions are equivalent if they represent the same number. See expression. The numerical expressions $3 + 2$ and $6 – 1$ are equivalent, since both are equal to 5. The algebraic expressions $3(x + 2)$ and $3x + 6$ are equivalent. For any value of the variable x, the expressions represent the same number.
evaluate	Evaluate refers to finding a numerical value. To evaluate an expression, replace each variable in the expression with a value and then calculate the value of the expression. To evaluate the numerical expression $3 + 4(5)$, we calculate $3 + 4(5) = 3 + 20 = 23$. To evaluate the variable expression $2x + 5$ when $x = 10$, we calculate $2x + 5 = 2(10) + 5 = 20 + 5 = 25$.
exponential notation	The exponential notation b^n (read as “ b to the power n ”) is used to express n factors of b. The number b is the base, and the number n is the exponent. $2^3 = 2 \bullet 2 \bullet 2 = 8$; The base is 2 and the exponent is 3. $3^2 \bullet 5^3 = 3 \bullet 3 \bullet 5 \bullet 5 \bullet 5 = 1{,}125$; The bases are 3 and 5. The exponents are 2 and 3.

Word or Phrase	Definition
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8 + x}{10}$, and $4v - w$.
greatest common factor	The greatest common factor (GCF) of two numbers is the greatest factor that divides the two numbers. The factors of 12 are 1, 2, 3, 4, 6, and 12. The factors of 18 are 1, 2, 3, 6, 9, and 18. Therefore the GCF of 12 and 18 is 6.
like terms	Terms of a mathematical expression that have the same variable part are referred to as like terms. See term. In the mathematical expression $2x + 6 + 3x + 5$, the terms $2x$ and $3x$ are like terms, and the terms 6 and 5 are like terms.
simplify	Simplify refers to converting a numerical or variable expression to a simpler form. A variable expression might be simplified by combining like terms. A fraction might be simplified by dividing numerator and denominator by a common divisor. $2x + 6 + 5x + 3 = 7x + 9$ $\frac{8}{12} = \frac{2}{3}$
square number	A square number, or perfect square, is a number that is a square of a natural number. The area of a square with side-lengths that are natural numbers is a square number. The square numbers are $1 = 1^2$, $4 = 2^2$, $9 = 3^2$, $16 = 4^2$, $25 = 5^2$, … .
terms	The terms in a mathematical expression involving addition (or subtraction) are the quantities being added (or subtracted). Terms that have the same variable part are referred to as like terms. The expression $2x + 6 + 3x + 5$ has four terms: $2x$, 6, $3x$, and 5. The terms $2x$ and $3x$ are like terms, since each is a constant multiple of x. The terms 6 and 5 are like terms, since each is a constant.
variable	A variable is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to quantities that vary in a relationship (as in a formula or an input-output rule). They may refer to unknown quantities in expressions, equations or inequalities. Finally, they may be used to generalize rules of arithmetic. In the equation $d = rt$, the quantities d, r, and t are variables. In the equation $2x = 10$, the variable x may be referred to as the unknown. The equation $a + b = b + a$ generalizes the commutative property of addition for all numbers a and b.

The Distributive Property

The distributive property relates the operations of multiplication and addition. The term “distributive” arises because the property is used to distribute the factor outside the parentheses over the terms inside the parentheses.

Suppose you earn \$9.00 per hour. If you work 3 hours on Saturday and 4 hours on Sunday, one way to compute your earnings is to compute your wages for each day and then add them. Another way is to multiply the hourly wage by the total number of hours. This example illustrates the distributive property.

$$(9 \times 3) + (9 \times 4) = 9(3 + 4)$$
$$27 + 36 = 9(7)$$

Order of Operations

There are many mathematical conventions that enable us to interpret mathematical notation and to communicate efficiently. The agreed-upon rules for interpreting mathematical notation, important for simplifying arithmetic and algebraic expressions, are called the standard order of operations.

Step 1: Do the operations in grouping symbols first (e.g., use rules 2-4 inside parentheses).
Step 2: Calculate all the expressions with exponents.
Step 3: Multiply and divide in order from left to right.
Step 4: Add and subtract in order from left to right.

Example: $$\frac{3^2 + (6 \bullet 2 - 1)}{5} = \frac{3^2 + (12 - 1)}{5} = \frac{3^2 + (11)}{5} = \frac{9 + (11)}{5} = \frac{20}{5} = 4$$

There are many times when these rules make complete sense and are quite natural. Take this case, for example:

You purchase 2 bottles of water for \$1.50 each and 3 bags of peanuts for \$1.25 each. Write an expression for this situation, and simplify the expression to find the total cost.

Expression: $$\underbrace{2 \bullet (1.50)}_{3.00} + \underbrace{3 \bullet (1.25)}_{3.75} = 6.75$$

In this problem, it is natural to find the cost of the 2 bottles of water and then the cost of the 3 bags of peanuts prior to adding these amounts together. In other words, we perform the multiplication operations before the addition operation.

However, if we were to perform the operations in order from left to right (as we read the English language from left to right), we would obtain a different result:

$$2(1.50) = 3 \longrightarrow 3 + 3 = 6 \longrightarrow 6(1.25) = 7.50\text{, wrong answer!}$$

Using Order of Operations to Simplify Expressions

Order of Operations	Expression	Comments
	$2^3 \div 2(5-2)$ $4 + 2 \bullet 10$	
1. Simplify expressions within grouping symbols.	$2^3 \div 2(3)$ $4 + 2 \bullet 10$	Parentheses are grouping symbols: Therefore, 5 – 2 = 3. The fraction bar is also a grouping symbol, so the first step here is to simplify the numerator and denominator.
2. Calculate powers.	$\frac{8 \div 2(3)}{4 + 2 \bullet 10}$	$2^3 = 2 \bullet 2 \bullet 2 = 8$
3. Perform multiplication and division from left to right.	$\frac{12}{4 + 20}$	In the numerator: Divide 8 by 2, then multiply by 3. In the denominator: Multiply 2 by 10.
4. Perform addition and subtraction from left to right.	$\frac{12}{24} = \frac{1}{2}$	Perform the addition: 4 + 20 = 24. Now the groupings in both the numerator and denominator have been simplified, so the final division can be performed.

Writing Expressions

The notation used for algebra is sometimes different from the notation used for arithmetic. For example:

- 54 means the sum of five tens and four ones, that is, 5(10) + 4.
- $5\frac{1}{2}$ means the sum of five and one-half. that is, $5 + \frac{1}{2}$.
- $5x$ means the product of 5 and x, which can also be written $5(x)$ or $5 \bullet x$. We typically do not write $5 \times x$ because the multiplication symbol '×' is easily confused with the variable x.

Evaluate or Simplify?

We use the word “evaluate” when we want to calculate the value of an expression.

Example: To evaluate 16 – 4(2), follow the rules for order of operations and compute:
16 – 4(2) = 16 – 8 = 8.

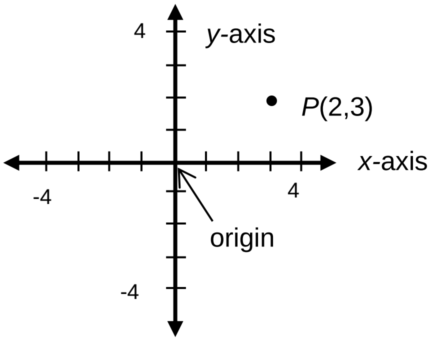
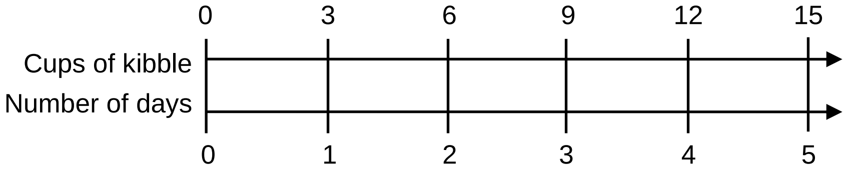
To evaluate $6 + 3x$ when $x = 2$, substitute 2 for x and calculate:
6 + 3(2) = 6 + 6 = 12.

We use the word “simplify” when rewriting a number or an expression in a form more easily readable or understandable.

Example: To simplify $2x + 3 + 5x$, combine like terms: $2x + 3 + 5x = 7x + 3$.

Sometimes it may not be clear what the simplest form of an expression is. For instance, by the distributive property, $4(x + 2) = 4x + 8$. For some applications, $4(x + 2)$ may be considered simpler than $4x + 8$, but for other applications, $4x + 8$ may be considered simpler than $4(x + 2)$.

STUDENT RESOURCES

Word or Phrase	Definition
coefficient	A coefficient is a number or constant factor in a term of an algebraic expression. In the expression $3x + 5$, 3 is the coefficient of the term $3x$, and 5 is the constant term.
coordinate plane	A coordinate plane is a plane with two perpendicular number lines (coordinate axes) meeting at a point (the origin). Each point P of the coordinate plane corresponds to an ordered pair (a, b) of numbers, called the coordinates of P. The point P may be denoted $P(a, b)$. The coordinate axes are often referred to as the x-axis and the y-axis respectively. The origin has coordinates (0, 0). 
dependent variable	A dependent variable is a variable whose value is determined by the values of the independent variables. See independent variable.
double number line	A double number line is a diagram made up of two parallel number lines that visually depict the relative sizes of two quantities. Double number lines are often used when the two quantities have different units, such as miles and hours. The proportional relationship “Wrigley eats 3 cups of kibble per day” can be represented in the following double number line diagram. 
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8 + x}{10}$, and $4v – w$.

<table>
<tr><th>Word or Phrase</th><th>Definition</th></tr>
<tr><td>independent variable</td><td>An independent variable is a variable whose value may be specified. Once specified, the values of the independent variables determine the values of the dependent variables.
For the equation $y = 3x$, y is the dependent variable and x is the independent variable. We may assign a value to x. The value assigned to x determines the value of y.</td></tr>
<tr><td>input-output rule</td><td>An input-output rule for a sequence of values is a rule that establishes explicitly an output value for each given input value.
<table><tr><td>**input value (x)**</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>x</td></tr><tr><td>**output value (y)**</td><td>1.5</td><td>3</td><td>4.5</td><td>6</td><td>7.5</td><td>$1.5x$</td></tr></table>
In the table above, the input-output rule could be $y = 1.5x$. In other words, to get the output value, multiply the input value by 1.5. If $x = 100$, then $y = 1.5(100) = 150$.</td></tr>
<tr><td>rate</td><td>See unit rate.</td></tr>
<tr><td>unit price</td><td>A unit price is a price for one unit of measure.
If 4 apples cost $1.00, then the unit price is $\frac{\$1.00}{4} = \$0.25$ for one apple, or 0.25 dollars per apple or 25 cents per apple.</td></tr>
<tr><td>unit rate</td><td>The unit rate associated with a ratio $a : b$ of two quantities a and b, $b \neq 0$, is the number $\frac{a}{b}$, to which units may be attached. This is sometimes referred to as the value of the ratio.
The ratio of 40 miles for every 5 hours has a unit rate of $\frac{40}{5} = 8$ miles per hour.</td></tr>
<tr><td>variable</td><td>A variable is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to quantities that vary in a relationship (as in a formula or an input-output rule). They may refer to unknown quantities in expressions, equations, or inequalities. Finally, they may be used to generalize rules of arithmetic.
In the equation $d = rt$, the quantities d, r, and t are variables.
In the equation $2x = 10$, the variable x may be referred to as the unknown.
The equation $a + b = b + a$ generalizes the commutative property of addition for all numbers a and b.</td></tr>
</table>

The Coordinate Plane

A coordinate plane is determined by a horizontal number line (the *x*-axis) and a vertical number line (the *y*-axis) intersecting at the zero on each line. The point of intersection (0, 0) of the two lines is called the origin. Points are located using ordered pairs (*x*, *y*).

- The first number (*x*-coordinate) indicates how far the point is to the right of the *y*-axis.
- The second number (*y*-coordinate) indicates how far the point is above the *x*-axis.

Point, coordinates, and interpretation

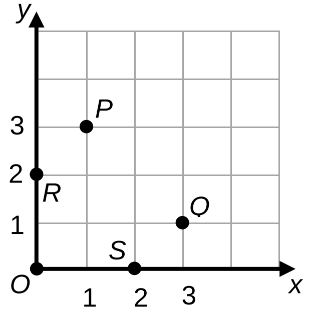
O (0, 0) → at the intersection of the axes

P (1, 3) → start at the origin, move 1 unit right, then 3 units up

Q (3, 1) → start at the origin, move 3 units right, then 1 unit up

R (0, 2) → start at the origin, move 0 units right, then 2 units up

S (2, 0) → start at the origin, move 2 units right, then 0 units up



Multiple Representations: Tables, Graphs, and Equations

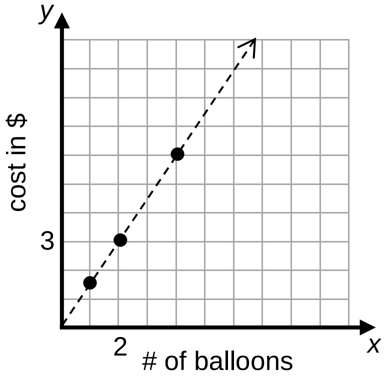
Suppose 4 balloons cost $6.00 and each balloon is the same price. Here are some representations for this relationship.

Table

Number of Balloons	Cost in $
4	6.00
2	3.00
1	1.50
8	12.00

Note that the unit price is $1.50 per balloon

Graph



Numbers of balloons must be discrete values (specifically, whole numbers), however a trend line may be drawn to show a growth pattern.

Equation (input-output rule)

Let y = cost in dollars and x = number of balloons.

We can see from the table that the unit price is 1.50 dollars per balloon.

It appears that multiplying any input value by 1.5 yields its corresponding output value.

Therefore, $y = 1.5x$.

STUDENT RESOURCES

Word or Phrase	Definition
distributive property	The distributive property states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c. $3(4 + 5) = 3(4) + 3(5)$ and $(4 + 5)8 = 4(8) + 5(8)$
equation	An equation is a mathematical statement that asserts the equality of two expressions. $18 = 8 + 10$ is an equation that involves only numbers. This is a numerical equation. $18 = x + 10$ is an equation that involves numbers and a variable and $y = x + 10$ is an equation that involves a number and two variables. These are both algebraic (variable) equations.
equivalent expressions	Two mathematical expressions are equivalent if, for any possible substitution of values for the variables, the two resulting numbers are equal. In particular, two numerical expressions are equivalent if they represent the same number. See expression. The numerical expressions $3 + 2$ and $9 – 4$ are equivalent, since both are equal to 5. The algebraic expressions $3(x + 4)$ and $3x + 12$ are equivalent. For any value of the variable x, the expressions represent the same number.
evaluate	Evaluate refers to finding a numerical value. To evaluate an expression, replace each variable in the expression with a value and then calculate the value of the expression. To evaluate the numerical expression $3 + 4(5)$, we calculate $3 + 4(5) = 3 + 20 = 23$. To evaluate the variable expression $2x + 5$ when $x = 10$, we calculate $2x + 5 = 2(10) + 5 = 20 + 5 = 25$.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Some mathematical expressions are 19, $7x$, $a + b$, $\frac{8 + x}{10}$, and $4v – w$.
inequality	An inequality is a mathematical statement that asserts the relative size or order of two objects. When the expressions involve variables, a solution to the inequality consists of values for the variables which, when substituted, make the inequality true. $5 > 3$ is an inequality. $x + 3 > 4$ is an inequality. All values for x that are greater than 1 are solutions to this inequality.

Word or Phrase	Definition
simplify	Simplify refers to converting a numerical or variable expression to a simpler form. A variable expression might be simplified by combining like terms. A fraction might be simplified by dividing numerator and denominator by a common divisor. $\frac{8}{12}$ may be simplified to the equivalent numerical expression $\frac{2}{3}$. $2x + 6 + 5x + 3$ may be simplified to the equivalent variable expression $7x + 9$.
solution to an equation	A solution to an equation involving variables consists of values for the variables which, when substituted, make the equation true. The value $x = 8$ is a solution to the equation $10 + x = 18$. If we substitute 8 for x in the equation, the equation becomes true: $10 + 8 = 18$.
solve an equation	To solve an equation refers to finding all values for the variables in the equation that, when substituted, make the equation true. Values that make an equation true are called solutions to the equation. To solve the equation $2x = 6$, one might think “two times what number is equal to 6?” Since $2(3) = 6$, the only value for x that satisfies this condition is 3. Therefore 3 is the solution.
substitution	Substitution refers to replacing a value or quantity with an equivalent value or quantity. If $x + y = 10$, and $y = 8$, then we may substitute this value for y in the equation to get $x + 8 = 10$.
tape diagram	A tape diagram is a graphical representation that uses length to represent relationships between quantities. We draw rectangles with a common width to represent quantities, and rectangles with the same length to represent equal quantities. Tape diagrams are typically used to represent quantities expressed in the same unit. This tape diagram represents a drink mixture with 3 parts grape juice for every 2 parts water. G G G W W

Variables in Algebra

Loosely speaking, variables are quantities that can vary. Variables are represented by letters or symbols. Variables have many different uses in mathematics. The use of variables, together with the rules of arithmetic, makes algebra a powerful tool. Three important ways that variables appear in algebra are the following.

Usage	**Examples**
Variables can represent an *unknown quantity* in an equation or inequality. In this case, the equation or inequality is valid only for specific value(s) of the variable.	$x + 4 = 9$ $5n = 20$ $y < 6$
Variables can represent *quantities that vary* in a relationship. In this case, there is always more than one variable in the equation.	Formula: $P = 2\ell + 2w$, $A = s^2$ Function (input-output rule): $y = 5x$, $y = x + 3$
Variables can represent *quantities in statements that generalize* rules of arithmetic. In this case, there may be one or more variables.	Commutative property of addition: $x + y = y + x$ Distributive property: $x(y + z) = xy + xz$

Using Shapes to Represent Variables

If the same shape (variable) is used more than once in an equation, it must represent the same value each place it appears. Two different shapes (variables) in an equation may represent the same value or different values.

This is allowed	This is allowed	This is NOT allowed
□ + □ = ○	□ + ⬯ = ◇	○ + ○ = △
7 + 7 = 14	6 + 6 = 12	6 + 4 = 10

Evaluate or Simplify?

We use the word “evaluate” when we want to calculate the value of an expression.

To evaluate 16 – 4(2), follow the rules for order of operations and compute: 16 – 4(2) = 16 – 8 = 8.

To evaluate $6 + 3x$ when $x = 2$, substitute 2 for x and calculate: $6 + 3(2) = 6 + 6 = 12$.

We use the word “simplify” when rewriting a number or an expression in a form more easily readable or understandable.

To simplify $2x + 3 + 5x$, combine like terms: $2x + 3 + 5x = 7x + 3$.

Sometimes it may not be clear what is the simplest form of an expression. For instance, by the distributive property, $4(x + 2) = 4x + 8$. For some applications, $4(x + 2)$ may be considered simpler than $4x + 8$, but for other applications, $4x + 8$ may be considered simpler than $4(x + 2)$.

How to Determine if an Equation is True

THE PIZZA PLACE MENU
(The variable represents the cost of an item.)

Pizza		**Drinks**	
Cheese slice (*c*)	\$1.00	Small drink (*s*)	\$0.95
Pepperoni slice (*p*)	\$1.25	Large drink (*L*)	\$1.75

What value from the menu above makes this equation true?

$$p + \boxed{} = 3c$$

Substitute:

$$1.25 + \boxed{} = 3(1.00)$$

$1.25 + 1.00 = 3.00$? NO

$1.25 + 1.25 = 3.00$? NO

$1.25 + 0.95 = 3.00$? NO

$1.25 + 1.75 = 3.00$? YES

The equation is true when $\boxed{}$ represents the cost of a large drink ($L = 1.75$).

Solving Equations Using a Mental Math and Substitution Strategy

To solve an equation using mental math and substitution, apply your knowledge of arithmetic facts to find a value for the unknown that makes the equation true.

Example 1: $8x = 40$

Think: *8 times what number is 40?*

Since $8(5) = 40$, $x = 5$

Check: $8(5) = 40$
$40 = 40$

Example 2: $8 + h = 20$

Think: *8 plus what number equals 20?*

Since $8 + 12 = 20$, $h = 12$

Check: $8 + 12 = 20$
$20 = 20$

Example 3: $4 = 12 - k$

Think: *4 is equal to 12 minus what number?*

Since $4 = 12 - 8$, $k = 8$

Check: $4 = 12 - 8$
$4 = 4$

Example 4: $\frac{n}{3} = 8$

Think: *What number divided by 3 is 8?*

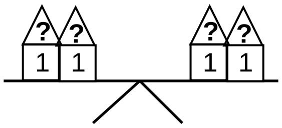
Since $\frac{24}{3} = 8$, $n = 24$

Check: $\frac{24}{3} = 8$
$8 = 8$

Balance Scales and Laws of Equality

Balance scales are physical representations of equations because both sides of a balanced scale must have the same weight, and both sides of an equation must have the same value.

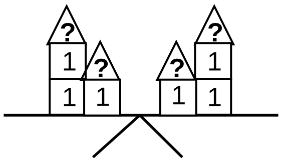
Imagine that each $\boxed{1}$ represents one unit of weight and each ? represents an unknown weight (not equal to zero). To represent unknowns, a popular variable is *x*.



The balanced scale above represents the equation $2x + 2 = 2x + 2$.

Example 1: Start with the balance scale above. Add the same thing to both sides, like 1.

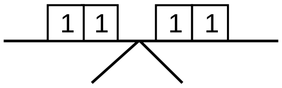
New scale:
(still balanced)



New equation: $2x + 2 + 1 = 2x + 2 + 1$
$2x + 3 = 2x + 3$

Example 2: Start with the balance scale above. Subtract the same thing from both sides, like 2*x*.

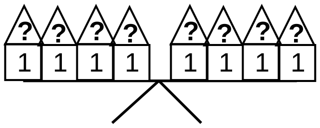
New scale:
(still balanced)



New equation: $2x + 2 – 2x = 2x + 2 – 2x$
$2 = 2$

Example 3: Multiply both sides by the same thing, like 2.

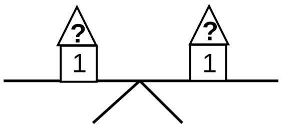
New scale:
(still balanced)



New equation: $2(2x + 2) = 2(2x + 2)$
$4x + 4 = 4x + 4$

Example 4: Divide both sides by the same thing, like 2. Here we are halving the weight on each side.

New scale:
(still balanced)



New equation: $\frac{2x+2}{2} = \frac{2x+2}{2}$
$x + 1 = x + 1$

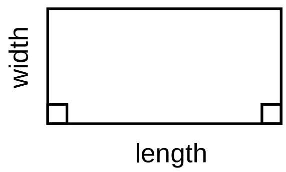
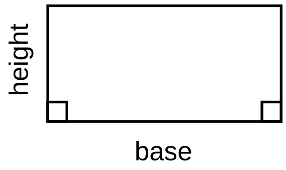
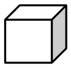
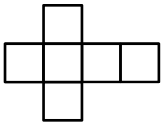
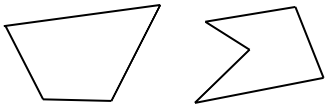
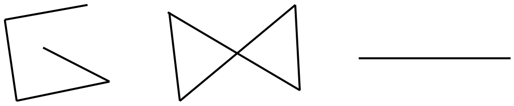
The addition property of equality states that if $a = b$ and $c = d$, then $a + c = b + d$. In other words, equals added to equals are equal. (See example 1 above.)

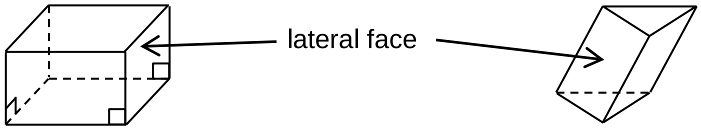
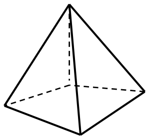
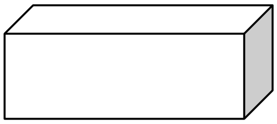
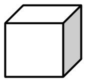
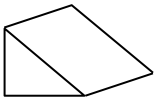
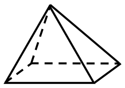
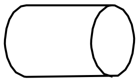
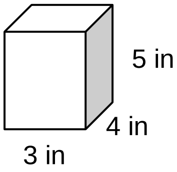
Note that this property extends to subtraction as well. (See example 2 above.)

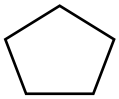
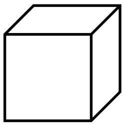
The multiplication property of equality states that if $a = b$ and $c = d$, then $ac = bd$. In other words, equals multiplied by equals are equal. (See example 3 above.)

Note that this property extends to division as well. (See example 4 above.)

STUDENT RESOURCES

Word or Phrase	Definition
area	The area of a two-dimensional figure is a measure of the size of the figure, expressed in square units. The area of a rectangle is the product of its length and width (Area = length • width). or The area of a rectangle is the product of its base and height (Area = base • height). If this rectangle has a length of 12 inches and a width of 5 inches, then: $A = \ell w$ $A = (12)(5)$ $A = 60$ square inches or $A = bh$ $A = (12)(5)$ $A = 60$ square inches  
net	A net for a three-dimensional figure is a two-dimensional pattern for the figure. If cut from a sheet of paper, for example, a net forms one connected piece which can be folded with the edges joined to form the given figure. cube  net of a cube 
plane	A plane is a flat, two-dimensional surface without holes that extends to infinity in all directions.
polygon	A polygon is a special kind of figure in a plane made up of a chain of line segments laid end-to-end to enclose a region.  polygons  not polygons

Word or Phrase	Definition
prism	*A <u>prism</u> is a solid figure in which two faces (the bases) are identical parallel polygons, and the other faces (referred to as the lateral faces) are parallelograms.* *If the lateral faces are perpendicular to the bases, the prism is a right prism. Otherwise, the prism is an oblique prism.*  A right rectangular prism is a right prism whose bases are rectangles and whose faces are rectangles. An oblique triangular prism is a prism whose bases are triangles and whose faces are parallelograms.
pyramid	A <u>pyramid</u> is a solid figure in which one face (the base) is a polygon, and the other faces (referred to as lateral faces) are triangles with a common vertex (referred to as the apex). The Egyptian pyramids are square pyramids since they have square bases. 
right rectangular prism	A <u>right rectangular prism</u> is a six-sided solid figure in which all the faces are rectangles. A rectangular box is a right rectangular prism. 
solid figure	A <u>solid figure</u> refers to a figure in three-dimensional space such as a prism or a cylinder.  cube  triangular prism  rectangular pyramid  cylinder
surface area	The <u>surface area</u> of a three-dimensional figure is a measure of the size of the surface of the figure, expressed in square units. If the surface of the three-dimensional figure consists of two-dimensional polygons, the surface area is the sum of the areas of the polygons. If this rectangular box has a length of 3 inches, a width of 4 inches, and a height of 5 inches, then  $SA = 2(\ell w) + 2(\ell h) + (wh)$ $SA = 2(3 \bullet 4) + 2(3 \bullet 5) + 2(4 \bullet 5)$ $SA = 94$ square inches

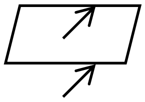
Word or Phrase	Definition
vertex	A vertex (plural of vertices) of a polygon or solid figure is a point where two edges meet. See polygon, solid figure. A pentagon has five vertices. 
volume	The volume of a three-dimensional figure is a measure of the size of the figure, expressed in cubic units. The volume of a right rectangular prism is the product of its length, width, and height. If this cube has a side length of 3 units, then $V = \ell wh$ $V = 3 \bullet 3 \bullet 3$ V = 27 cubic inches 

Base of a Polygon (*b*) Versus Base of a Solid Figure (*B*)

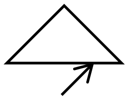
The base of a polygon is a predesignated side of the figure. It is typically denoted with a "*b*."

The base is usually regarded as the "bottom" of the polygon. The top is also a base, if it is parallel to the bottom.

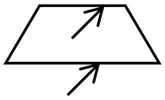
Any side of a parallelogram may be the base.



Any side of a triangle may be chosen as the base.



A trapezoid has two bases. They are the parallel sides.

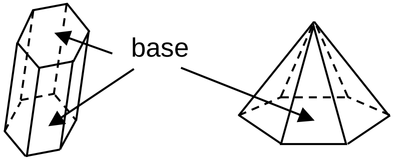


The base of a solid figure is a predesignated face of the figure. It is typically denoted with a *"B*."

The base is usually regarded as the "bottom" of the figure, on which it is standing. The "top" of a figure is sometimes also referred to as a base if it is identical and parallel to the "bottom."

This right prism has two parallel bases (hexagons).

This right pyramid has two one base (a hexagon).



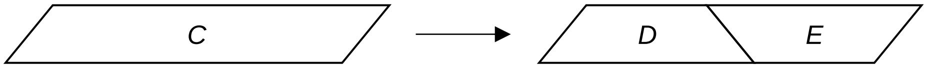
Composing and Decomposing Shapes: “Cut-Up Strategies”

Composing shapes refers to joining geometric shapes without overlaps to form other shapes.

Here are two identical triangles (*A* and *B*). When joining *A* and *B* (after rotating *B*), the result is a parallelogram with twice the area of each of the given triangles.

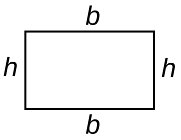
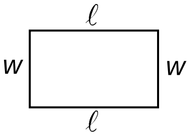
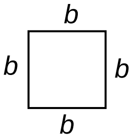
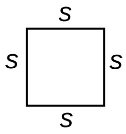
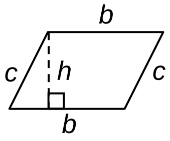
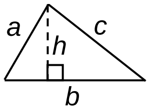
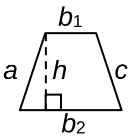
Decomposing shapes refers to taking a given geometric shape, and identifying geometric shapes that meet without overlap to form that given shape.

Given parallelogram *C*, we can identify a segment that creates two identical trapezoids *D* and *E*, each with one-half the area of *C*.



Composing and decomposing shapes are useful strategies for finding area formulas for common polygons derived from ones we already know. We refer to these methods collectively as “cut-up” strategies. For example, first we learn the formula for area of a rectangle. Then we can use a cut-up strategy to find the formula for area of a parallelogram. Then we can use other cut-up strategies to find the formulas for area of a triangle and area of a trapezoid.

Summary of Area Formulas

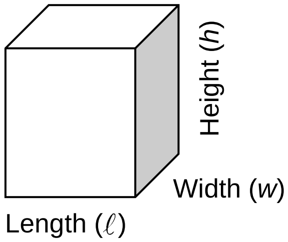
Shape/Definition	Diagram	Area
Rectangle a quadrilateral with 4 right angles	 	$A = bh$ or $A = \ell w$
Square a rectangle with 4 sides of equal length	 	$A = b^2$ or $A = s^2$
Parallelogram a quadrilateral with opposite sides parallel		$A = bh$
Triangle a polygon with three sides		$A = \frac{1}{2}bh$
Trapezoid a quadrilateral with at least one pair of parallel sides		$A = \frac{1}{2}(b_1 + b_2)h$

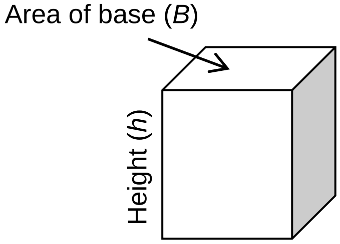
Volume and Surface Area of Right Rectangular Prisms

A right rectangular prism is identified by its length, width, and height.

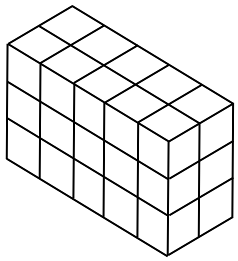
The area of the base is the product of the length and width ($B = \ell w$).

If the top and bottom rectangular faces are chosen as the bases, then the other rectangles are referred to as the lateral faces.

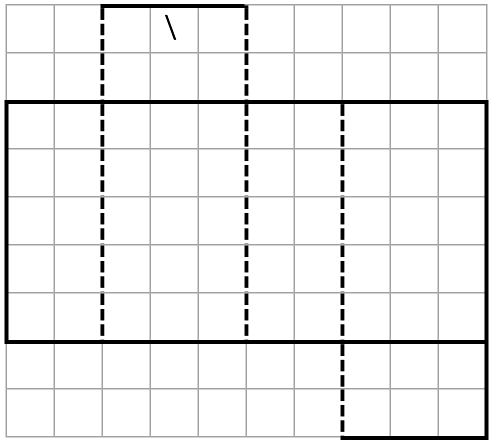




Right rectangular prism made with 30 cubes



Net of the same prism



Volume

The volume of a prism may be computed by counting layers of unit cubes. In the prism above, each layer has 10 cubes (5 × 2). There are 3 layers.

The volume is $(5 \times 2)(3) = 10(3) = 30$ cubic units.

In general, multiply the area of the base (B) by the height.

$V = \ell wh$ OR $V = Bh$

Surface Area

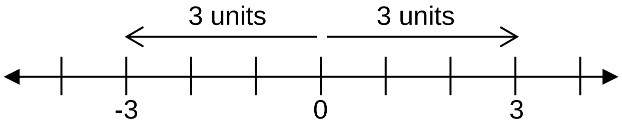
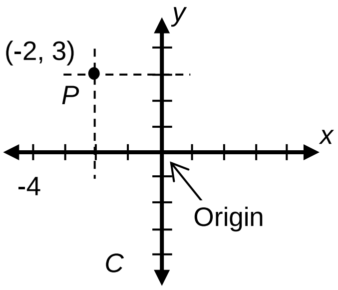
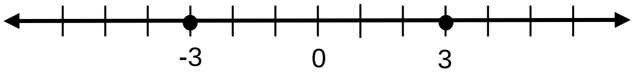
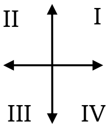
The surface area may be computed by creating a net that shows the areas of each face of the prism. In this prism there are two faces with dimensions 2 × 5, two faces with dimensions 3 × 2, and two faces with dimensions 3 × 5.

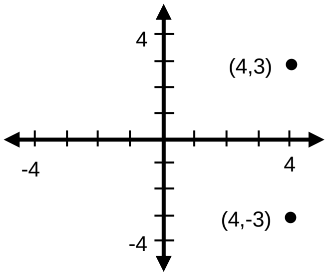
The surface area is
$2(2 \times 5) + 2(3 \times 2) + 2(3 \times 5)$
$= 20 + 12 + 30$
$= 62$ square units.

In general, find the area of each rectangular face.

$SA = \ell w + \ell w + wh + wh + \ell h + \ell h$ OR
$SA = 2\ell w + 2wh + 2\ell h$ OR
$SA = 2(\ell w + wh + \ell h)$

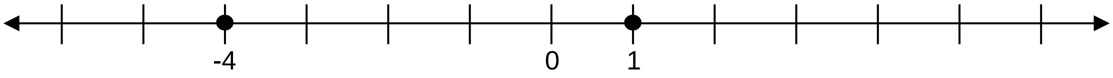
STUDENT RESOURCES

Word or Phrase	Definition
absolute value	The absolute value $\lvert x \rvert$ of a number x is the distance from x to 0 on the number line. $\lvert 3 \rvert = 3$ and $\lvert -3 \rvert = 3$, because both 3 and -3 are 3 units from 0 on the number line. 
coordinate plane	A coordinate plane is a plane with two perpendicular number lines (coordinate axes) meeting at a point (the origin). Each point *P* of the coordinate plane corresponds to an ordered pair (*a*, *b*) of numbers, called the coordinates of *P*. The point *P* may be denoted *P* (*a*, *b*). The coordinate axes are often referred to as the *x*-axis and the *y*-axis respectively. The origin has coordinates (0, 0). The *x*-coordinate of *P* is -2, and the *y*-coordinate of *P* is 3. Point *P* (-2, 3) is an ordered pair. 
integers	The integers are the whole numbers and their opposites. They are the numbers 0, 1, 2, 3, … and -1, -2, -3, … .
opposite of a number	The opposite of a number *n*, written -*n*, is its additive inverse. Algebraically, the sum of a number and its opposite is zero. Geometrically, the opposite of a number is the number on the other side of zero at the same distance from zero.  The opposite of 3 is -3, because 3 + (-3) = -3 + 3 = 0. The opposite of -3 is -(-3) = 3. Thus, the opposite of a number does not have to be negative.
quadrants	The coordinate axes of a coordinate plane separate the plane into four regions, called quadrants. The quadrants are labeled I – IV starting from the upper right region and going counterclockwise. 

Word or Phrase	Definition
reflection	The <u>reflection</u> of a plane through a line refers to the transformation that takes a point on one side of the line to its mirror image on the other side of the line. When the plane is reflected through the *x*-axis, the point (4,3) is taken to the point (4,-3). 

Integers on the Number Line

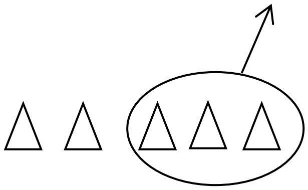
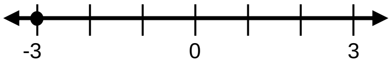
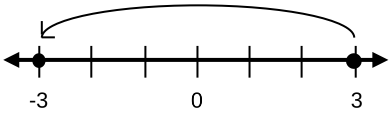
<u>Integers</u> (whole numbers and their opposites) may be represented on a horizontal or vertical number line.



On the number line above, points are graphed at -4 and 1. While 0 is labeled, it is not graphed.

Two Uses of the Minus Sign

Here are two ways to interpret the minus sign, along with some examples.

When the minus sign is between two expressions, it means "subtract the second expression from the first."	Example: 5 – 3 The phrase "5 minus 3" can be read: • 5 take away 3 • The difference between 5 and 3 • Subtract 3 from 5 
In front of a number, a minus sign can mean "negative" or "opposite." -3 is 3 units less than zero on the number line. -3 is also the opposite of 3. "Minus" can be thought of as a reflection or mirror image. In this case, we are reflecting the number line through zero.	Example: -3 The phrase "minus 3" can be read: • Negative 3  Pictorially, this is a location on the number line that is 3 units left of zero. • Opposite of 3  This is the value you get by first locating 3 on the number line, and then locating that same distance on the opposite side of zero. Geometrically, minus can be thought of as a reflection or mirror image. In this case, the reflection of 3 through zero is -3.

Distance and Absolute Value

The absolute value of a number is its distance from zero on the number line.

A distance 25 units in the positive direction from zero is written $|+25| = 25$.

A distance 25 units in the negative direction from zero is written $|-25| = 25$.

The absolute value of a positive number is equal to the number itself. The absolute value of a negative number is the opposite of the number. The absolute value of zero is simply zero.

Distance is always greater than or equal to zero.

Elevation relative to sea level is measured vertically from sea level. Sea level is typically represented as elevation = 0. Therefore, elevation may be positive, negative, or zero.

The vertical number line below represents some people and animals at elevations from 25 meters below sea level (-25 m) to 25 meters above sea level (+25 m).

What	**Elevation**	**Distance from zero (sea level)**	**Absolute value equation for Distance from sea level**
crow	+25 m	25 m	$|25| = 25$
gull	+15 m	15 m	$|15| = 15$
swimmer	0 m	0 m	$|0| = 0$
dolphin	-25 m	25 m	$|-25| = 25$

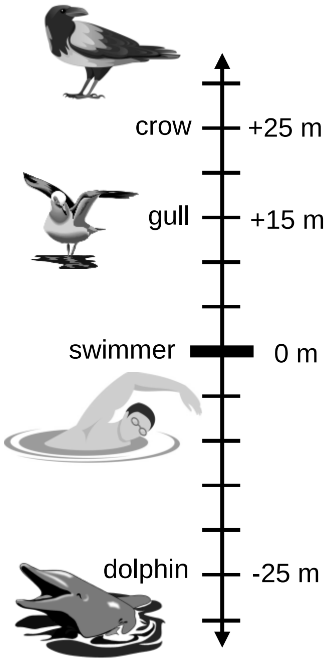
Here are some true statements about elevation:

- The gull is at a higher elevation than the dolphin: $15 > -25$
- The swimmer is at a lower elevation than the crow: $0 < 25$

Here are some true statements about absolute value:

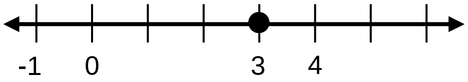
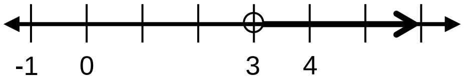
- The dolphin and the crow are the same distance from 0: $|-25| = |25|$
- The dolphin and the crow are both 25 meters from sea level: $|-25| = |25|$

The dolphin is farther from sea level than the gull: $|-25| > |15|$



Graphing Inequalities on the Number Line

Here are some ways to represent solutions to inequalities on a number line.

Values on the number line that satisfy the equation $x = 3$.	Values on the number line that satisfy the inequality $x > 3$.
	
The dots represents that a number is graphed.	The "open dot" indicates that $x = 3$ is not included in the solution set. That is, it is not a solution to the inequality $x > 3$. The arrow indicates that all numbers to the right on the number line are solutions.

The Coordinate Plane

A coordinate plane is determined by a horizontal number line (the *x*-axis) and a vertical number line (the *y*-axis) intersecting at the zero on each line. The point of intersection (0, 0) of the two lines is called the origin.

Points are located using ordered pairs (x, y).

- The first number (*x*-coordinate) indicates how far the point is to the right or left of the *y*-axis.
- The second number (*y*-coordinate) indicates how far the point is above or below the *x*-axis.

The axes (plural of axis) divide the plane into four regions, called quadrants. By convention, we number the quadrants using Roman numerals I-IV, starting with the upper right quadrant (first quadrant) and moving counterclockwise to the lower right quadrant (fourth quadrant). The axes may be considered as boundary lines and are not part of any quadrant.

Point and Coordinates	Interpretation	Location
O (0, 0)	At the intersection of the axes.	origin
P (1, 3)	Start at the origin, move 1 unit right, then 3 units up.	Quadrant I
Q (2, -1)	Start at the origin, move 2 units right, then 1 unit down.	Quadrant IV
R (0, -2)	Start at the origin, move 0 units right or left, then 2 units down.	*y*-axis