Name______________________________ Period __________ Date _______________

GRADE 7
RESOURCE GUIDE
PART 1

MATHLINKS: GRADE 7
RESOURCE GUIDE: PART 1

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THE STANDARDS FOR MATHEMATICAL PRACTICE

In addition to the mathematical topics you will learn about in this course, your teacher will help you become better at what are called the Mathematical Practices. The Standards for Mathematical Practice describe a variety of processes and strategies to help you to be more mathematically proficient and fluent students.

One way to think about the practices is in groupings.

HABITS OF MIND

MP1 Make sense of problems and persevere in solving them
MP6 Attend to precision

REASONING AND EXPLAINING

MP2 Reason abstractly and quantitatively
MP3 Construct viable arguments and critique the reasoning of others

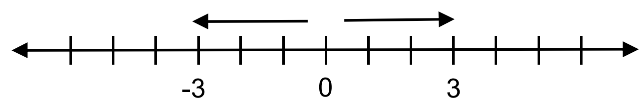
MODELING AND USING TOOLS

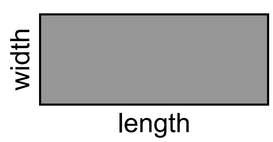
MP4 Model with mathematics
MP5 Use appropriate tools strategically

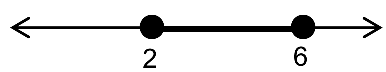
SEEING STRUCTURE AND GENERALIZING

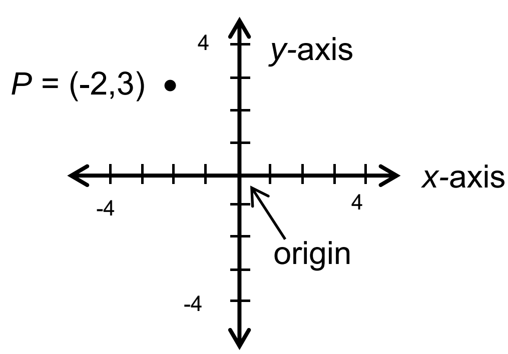
MP7 Look for and make use of structure
MP8 Look for and make use of repeated reasoning

WORD BANK

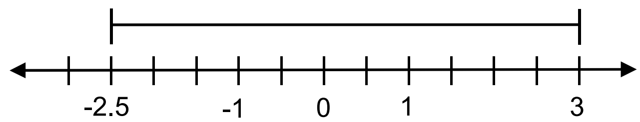
Word or Phrase	Definition
absolute value	The absolute value $\|x\|$ of a number x is the distance from x to 0 on the number line. Example: $\|3\| = 3$ and $\|-3\| = 3$, because both 3 and -3 are 3 units from 0 on the number line. 
addend	In an addition problem, an addend is a number to be added. See sum. Example: 7 (addend) + 5 (addend) = 12 (sum)
addition property of equality	The addition property of equality states that if $a = b$ and $c = d$, then $a + c = b + d$. In other words, equals added to equals are equal. Example: If $3 = 2 + 1$ and $5(2) = 10$, then $3 + 5(2) = 2 + 1 + 10$.
additive identity property	The additive identity property states that $a + 0 = 0 + a = a$ for any number a. In other words, the sum of a number and 0 is the number. We say that 0 is an additive identity. The additive identity property is sometimes called the addition property of zero. Example: $3 + 0 = 3$, $0 + 7 = 7$, $-5 + 0 = -5 = 0 + (-5)$
additive inverse	The additive inverse of a is the number b such that $a + b = b + a = 0$. The additive inverse of a is denoted by $-a$. Example: -4 is the additive inverse of 4. Example: 0.25 is the additive inverse of -0.25.

additive inverse property	The <u>additive inverse property</u> states that $a + (-a) = 0$ for any number a. In other words, the sum of a number and its opposite is 0. The number $-a$ is the additive inverse of a. Examples: $3 + (-3) = 0$, $-25 + 25 = 0$
algebraic expression	See <u>expression</u>.
algorithm	An <u>algorithm</u> is an organized procedure, or step-by-step recipe, for performing a calculation or finding a solution. Example: The traditional procedure for dividing whole numbers is called the <u>long division algorithm</u>.
area	The <u>area</u> of a two-dimensional figure is a measure of the size of the figure, expressed in square units. The <u>area of a rectangle</u> is the product of its length and its width.  Area = (length) × (width) Example: If a rectangle has a length of 12 inches and a width of 5 inches, its area is $5 \times 12 = 60$ square inches.
associative property of addition	The <u>associative property of addition</u> states that $a + (b + c) = (a + b) + c$ for any three numbers a, b, and c. In other words, the sum does not depend on the grouping of the addends. Example: $9 + (1 + 14) = (9 + 1) + 14$
associative property of multiplication	The <u>associative property of multiplication</u> states that $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for any three numbers a, b, and c. In other words, the product does not depend on the grouping of the factors. Example: $(3 \bullet 4) \bullet 5 = 3 \bullet (4 \bullet 5)$
axis	See <u>coordinate plane</u>.

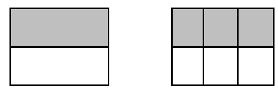
benchmark fraction	A <u>benchmark fraction</u> refers to a fraction that is easily recognizable. It is easily identified on the number line, and it is more commonly used in everyday experiences. Example: Some benchmark fractions are $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$.
boundary point of a solution set	A <u>boundary point of a solution set</u> is a point for which any segment surrounding it on the number line contains both solutions and non-solutions. If the solution set is an interval, the boundary points of the solution set are the endpoints of the interval. Example: The boundary point for $2 + x > 1$ (or $x > -1$) is $x = -1$. In this case the boundary point is NOT part of the solution set.  Example: The boundary points for $2 \le x \le 6$ are 2 and 6. In this case the boundary points ARE part of the solution set. 
coefficient	A <u>coefficient</u> is a number or constant factor in a term of an algebraic expression. Example: In the expression $3x + 5$, 3 is the coefficient of the linear term $3x$, and 5 is the constant coefficient.
commutative property of addition	The <u>commutative property of addition</u> states that $a + b = b + a$ for any two numbers a and b. In other words, changing the order of the addends does not change the sum. Example: $14 + 6 = 6 + 14$
commutative property of multiplication	The <u>commutative property of multiplication</u> states that $a \bullet b = b \bullet a$ for any two numbers a and b. In other words, changing the order of the factors does not change the product. Example: $3 \bullet 5 = 5 \bullet 3$

complex fraction	A complex fraction is a fraction whose numerator or denominator is a fraction. Example: $\frac{\frac{4}{5}}{\frac{1}{2}}$ and $\frac{\frac{1}{5}}{3}$ are complex fractions.
conjecture	A conjecture is a statement that is proposed to be true, but has not been proven to be true nor to be false. Example: After creating a table of sums of odd numbers such as $1 + 3 = 4$, $1 + 5 = 6$, $5 + 7 = 12$, $3 + 9 = 12$, etc., we may make a conjecture that the sum of any two odd numbers is an even number. This conjecture can be proven to be true.
consecutive integers	Two integers are consecutive if one of them is equal to the other plus 1. Example: The integers 5 and 6 are consecutive, since $6 = 5 + 1$.
constant term	A constant term in an algebraic expression is a term that has a fixed numerical value. Example: In the expression $5 + 12x^2 – 7$, the terms 5 and -7 are constant terms.
coordinate plane	A coordinate plane is a plane with two perpendicular number lines (coordinate axes) meeting at a point (the origin). Each point *P* of the coordinate plane corresponds to an ordered pair (*a*,*b*) of numbers, called the coordinates of *P*. Example: The coordinate axes are often referred to as the *x*-axis and the *y*-axis respectively. The *x*-coordinate *a* of *P* is the number where the line through *P* parallel to the *y*-axis hits the *x*-axis, and the *y*-coordinate *b* of *P* is the number where the line through *P* parallel to the *x*-axis hits the *y*-axis. Points on the *x*-axis have coordinates (*a*,0), and points on the *y*-axis have coordinates (0,*b*). The origin has coordinates (0,0). 

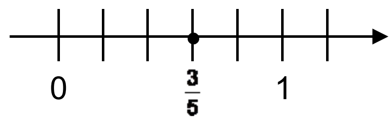
coordinates	The coordinates of a point in a coordinate plane are the values that show the location of a point. See coordinate plane. Example: The coordinates of the point $P = (5, 8)$ in the coordinate plane are the numbers 5 and 8, which indicate a position 5 units to the right and 8 units up from the origin.
data set	A data set is a collection of pieces of information, often numbers, obtained from observation, questioning, or measuring. Example: The following data set was collected in a survey of 10 students about the number of siblings they have: {2, 0, 2, 1, 4, 3, 1, 1, 2, 1}
decimal	A decimal is an expression of the form $n.abc\ldots$, where n is a whole number written in standard form, and $a, b, c, \ldots$ are digits. Each decimal represents a unique nonnegative real number and is referred to as a decimal expansion of the number. Example: The decimal expansion of $\frac{4}{3}$ is 1.333333… . Example: The decimal expansion of $\frac{5}{4}$ is 1.25 = 1.2500000… . Example: The decimal expansion of π is 3.14159… .
deductive reasoning	Deductive reasoning is a form of reasoning in which the conclusion is justified by an argument based on definitions, known facts, and accepted rules of logic. Example: A mathematical proof is a form of deductive reasoning.
denominator	The denominator of the fraction $\frac{a}{b}$ is b. Example: The denominator of $\frac{3}{7}$ is 7.
difference	In a subtraction problem, the difference is the result of subtraction. The minuend is the number from which another number is being subtracted, and the subtrahend is the number that is being subtracted. Example: $\underset{\text{minuend}}{12} - \underset{\text{subtrahend}}{4} = \underset{\text{difference}}{8}$

distance	The distance between two points on a number line is the absolute value of their difference. Example: The distance between 3 and -2.5 is $\|3 - (-2.5)\| = \|3 + 2.5\| = \|5.5\| = 5.5$ or $\|(-2.5) - 3\| = \|-5.5\| = 5.5$. 
distributive property	The distributive property states that $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for any three numbers a, b, and c. Example: $3(4 + 5) = 3(4) + 3(5)$ and $(4 + 5)8 = 4(8) + 5(8)$
divisible	A number n is divisible by q if n is a multiple of q. When we divide n by q, the remainder is 0. Example: 24 is divisible by 8 because $24 = 3 \bullet 8$.
division	Division is the mathematical operation that is inverse to multiplication. For $b \neq 0$, division by b is multiplication by the multiplicative inverse $\frac{1}{b}$ of b, $$a \div b = a \bullet \frac{1}{b}.$$ In this division problem, the number a to be divided is the dividend, the number b by which a is divided is the divisor, and the result $a \div b$ of the division is the quotient: $\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$ $\qquad$ $\text{divisor}\overline{\smash{)}\text{dividend}}$ with quotient above Other notations for $a \div b$ are $\frac{a}{b}$, a/b, and $b\overline{\smash{)}a}$. Example: "Twelve divided by 2" may be written $12 \div 2$, $\frac{12}{2}$, or $2\overline{\smash{)}12}$.

divisor	In a division problem, the <u>divisor</u> is the number by which another is divided. See <u>division</u>. Example: In $12 \div 3 = 4$, the divisor is 3.
empirical evidence	<u>Empirical evidence</u> is evidence that is based on experience or observation. Data collected from experimentation are called <u>empirical data</u>.
equation	An <u>equation</u> is a mathematical statement that asserts the equality of two expressions. When the equation involves variables, a <u>solution to the equation</u> consists of values for the variables which, when substituted, make the equation true. Example: $5 + 6 = 14 - 3$ is an equation that involves only numbers. Example: $10 + x = 18$ is an equation that involves numbers and a variable; the value for x must be 8 to make this equation true.
equivalent expressions	Two mathematical expressions are <u>equivalent</u> if for any possible substitution of values for the variables, the two resulting numbers are equal. In particular, two numerical expressions are equivalent if they represent the same number. See <u>expression</u>. Example: The algebraic expressions $3(x + 4)$ and $3x + 12$ are equivalent. For any value of the variable x, the expressions represent the same number. Example: The numerical expressions $3 + 2$ and $9 - 4$ are equivalent, since both are equal to 5.

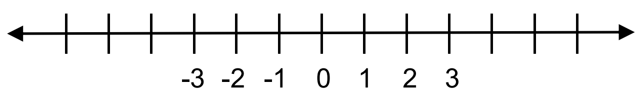
equivalent fractions	The fractions $\frac{a}{b}$ and $\frac{c}{d}$ are <u>equivalent</u> if they represent the same point on the number line. This occurs if the results of the division problems $a \div b$ and $c \div d$ are equal. Example: Since $\frac{1}{2} = 1 \div 2 = 0.5$ and $\frac{3}{6} = 3 \div 6 = 0.5$, the fractions $\frac{1}{2}$ and $\frac{3}{6}$ are equivalent.  $\frac{1}{2}$ $\frac{3}{6}$
equivalent ratios	Two ratios are <u>equivalent</u> if each number in one ratio is a multiple of the corresponding number in the other ratio by the same positive number. Example: The ratios "20 to 12" and "5 to 3" are equivalent because 20 and 12 respectively are multiples of 5 and 3 by the same number 4: $20 = 4 \cdot 5$, $12 = 4 \cdot 3$.
estimate	An <u>estimate</u> is an educated guess.
evaluate	<u>Evaluate</u> refers to finding a numerical value. To <u>evaluate an expression</u>, replace each variable in the expression with a value and then calculate the value of the expression. Example: To evaluate the expression $3 + 4(5)$, we calculate $3 + 4(5) = 3 + 20 = 23$. Example: To evaluate the expression $2x + 5$ when $x = 10$, we calculate $2x + 5 = 2(10) + 5 = 20 + 5 = 25$.
even number	A number is <u>even</u> if it is divisible by 2. Example: The even integers are 0, 2, 4, 6… and -2, -4, -6 … . Example: The even whole numbers are 0, 2, 4, 6, … .
event	An <u>event</u> is a subset of the sample space. See <u>sample space</u>. Example: In the probability experiment of rolling a number cube, "rolling an even number" is an event, because getting a 2, 4, or 6 is a subset (part) of the sample space of {1, 2, 3, 4, 5, 6}.

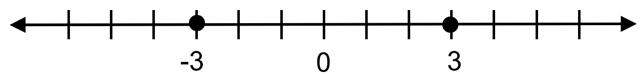
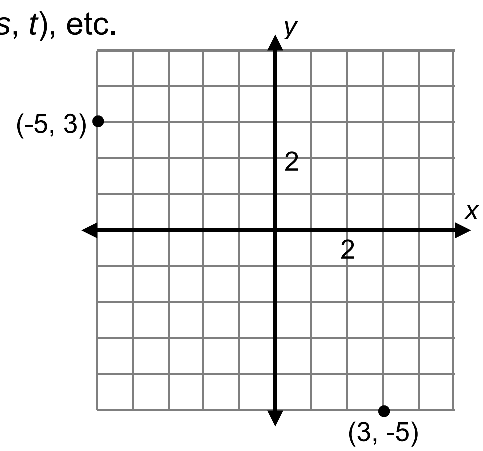
experimental probability	In a repeated probability experiment, the experimental probability of an event is the number of times the event occurs divided by the number of trials. Example: If, in 25 rolls of a number cube, we obtain an even number 11 times, we say that the experimental probability of rolling an even number is $\frac{11}{25} = 0.44 = 44\%$.
exponent	See exponential notation.
exponential notation	The exponential notation b^n (read as "b to the power n" or "b to the n^{th} power") is used to express n factors of b. The number b is the base, and the natural number n is the exponent. Example: $2^3 = 2 \bullet 2 \bullet 2 = 8$ The base is 2 and the exponent is 3. Example: $3^2 \bullet 5^3 = 3 \bullet 3 \bullet 5 \bullet 5 \bullet 5 = 1{,}125$ The bases are 3 and 5.
expression	A mathematical expression is a combination of numbers, variables, and operation symbols. When values are assigned to the variables, an expression represents a number. Example: Some mathematical expressions are $7x$, $a + b$, $4v - w$, and 19.
factor of a number	A factor of a number is a divisor of the number. See product. Example: The factors of 12 are 1, 2, 3, 4, 6, and 12.
fair game	A game of chance is a fair game if all players have equal probabilities of winning. Example: A two-person game of chance is a fair game if each player has probability one-half of winning, that is, if each player has the same probability of winning as of losing.

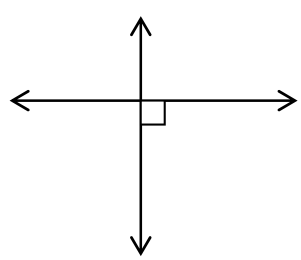
fraction	A fraction is a number expressible in the form $\frac{a}{b}$ where a is a whole number and b is a positive whole number. Example: The fraction $\frac{3}{5}$ is represented by the dot on the number line. 
frequency	Frequency refers to how often something occurs.
generalization	Generalization is the process of formulating general concepts by abstracting common properties from specific cases. Example: We may notice that the sum of two numbers is the same regardless of order (as in $3 + 5 = 5 + 3$). We generalize by writing the statement: $a + b = b + a$ for all numbers a and b.
greatest common factor	The greatest common factor (GCF) of two numbers is the greatest factor that divides the two numbers. Example: The factors of 12 are 1, 2, 3, 4, 6, and 12. The factors of 18 are 1, 2, 3, 6, 9, and 18. Therefore the GCF of 12 and 18 is 6.
inductive reasoning	Inductive reasoning is a form of reasoning in which the conclusion is supported by the evidence, but is not proved. Example: After creating a table of sums of odd numbers such as $1 + 3 = 4$, $1 + 5 = 6$, $5 + 7 = 12$, $3 + 9 = 12$, etc., we may reason inductively that the sum of any two odd numbers is an even number.

inequality	An inequality is a mathematical statement that asserts the relative size or order of two objects. When the expressions involve variables, a solution to the inequality consists of values for the variables which, when substituted, make the inequality true. Example: $5 > 3$ is an inequality. Example: $x + 3 > 4$ is an inequality. Its solution (which is also an inequality) is $x > 1$.
integers	The integers are the whole numbers and their opposites. They are the numbers 0, 1, 2, 3, … and -1, -2, -3, … .
inverse operation	The inverse operation to a mathematical operation reverses the effect of the operation. Example: Addition and subtraction are inverse operations. Example: Multiplication and division are inverse operations.
least common multiple	The least common multiple (LCM) of two numbers is the least number that is a multiple of both numbers. Example: The multiples of 8 are 8, 16, 24, 32, 40, ... The multiples of 12 are 12, 24, 36, 48, … . Therefore the LCM of 8 and 12 is 24.
linear inequality	A linear inequality in the variables x and y is a mathematical statement which can be written in the form $ax + by + c > 0$.
line segment	A line segment is a straight-line path joining two points. The line segment between two points P and Q consists of all points on the straight line through P and Q that lie between P and Q. The points P and Q are the endpoints of the line segment.
minuend	In a subtraction problem, the minuend is the number from which another is subtracted. See difference. Example: In $12 – 4 = 8$, the minuend is 12.

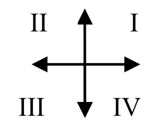
mixed number	A <u>mixed number</u> is an expression of the form $n\frac{p}{q}$, which is a shorthand for $n + \frac{p}{q}$, where n, p, and q are positive whole numbers. Example: The mixed number $4\frac{1}{4}$ ("four and one fourth") is shorthand for $4 + \frac{1}{4}$. It should not be confused with the product $4 \bullet \frac{1}{4} = 1$.
multiple	A <u>multiple</u> of a number m is a number of the form $k \bullet m$ for some number k. Example: The numbers 5, 10, 15, and 20 are multiples of 5, since $1 \bullet 5 = 5$, $2 \bullet 5 = 10$, $3 \bullet 5 = 15$, and $4 \bullet 5 = 20$.
multiplication property of equality	The <u>multiplication property of equality</u> states that if $a = b$ and $c = d$, then $ac = bd$. In other words, equals multiplied by equals are equal. Example: If $10 = 2 \bullet 5$ and $4 + 2 = 5 + 1$ then $10 \bullet (4 + 2) = 2 \bullet 5 \bullet (5 + 1)$ Check $60 = 60$
multiplication property of zero	The <u>multiplication property of zero</u> states that $a \bullet 0 = 0 \bullet a = 0$ for all numbers a. Example: $4 \bullet 0 = 0$, $0 \bullet (-5) = 0$
multiplicative identity property	The <u>multiplicative identity property</u> states that $a \bullet 1 = 1 \bullet a = a$ for all numbers a. In other words, 1 is a <u>multiplicative identity</u>. The multiplicative identity property is sometimes called the <u>multiplication property of 1</u>. Example: $4 \bullet 1 = 4$, $1 \bullet (-5) = -5$
multiplicative inverse	For $b \neq 0$, the <u>multiplicative inverse</u> of b is the number, denoted by $\frac{1}{b}$, that satisfies $b \bullet \frac{1}{b} = 1$. The multiplicative inverse of b is also referred to as the <u>reciprocal</u> of b. Example: The multiplicative inverse of 4 is $\frac{1}{4}$, since $4 \bullet \frac{1}{4} = 1$.

multiplicative inverse property	The <u>multiplicative inverse property</u> states that $a \bullet \frac{1}{a} = \frac{1}{a} \bullet a = 1$ for every number $a \neq 0$. See <u>multiplicative inverse</u>. Example: $25 \bullet \frac{1}{25} = \frac{1}{25} \bullet 25 = 1$
negative number	A <u>negative number</u> is a number that is less than zero (written "$a < 0$"). The negative numbers are the numbers to the left of 0 on a number line. Example: The numbers -2, -4.76, and $-\frac{1}{4}$ are negative. Example: The numbers 2 and 5.3 are not negative. They are positive. Example: The number 0 is neither negative nor positive.
number line	A <u>number line</u> is a visualization of the real numbers as a straight line. Usually tick marks are used to represent specific benchmark numbers such as 0 and 1. Example: 
numerator	The <u>numerator</u> of the fraction $\frac{a}{b}$ is a. Example: The numerator of $\frac{3}{7}$ is 3.
numerical data	<u>Numerical data</u> are data consisting of numbers.
numerical expression	A <u>numerical expression</u> is a mathematical expression that has only numbers and no variables. See <u>expression</u>. Example: 4, 5 + 8, 10.56, $\frac{11 + 3}{7}$
odd number	A number is <u>odd</u> if it is *not* divisible by 2. Example: The odd integers are 1, 3, 5,… and -1, -3, -5, … .

opposite of a number	The <u>opposite of a number</u> n, written $-n$, is its additive inverse. Algebraically, the sum of a number and its opposite is zero. Geometrically, the opposite of a number is its reflection through zero on the number line. Example: The opposite of 3 is -3, because $3 + (-3) = -3 + 3 = 0$. Example: The opposite of -3 is $-(-3) = 3$. Thus the opposite of a number does not have to be negative. 
order of operations	An <u>order of operations</u> is a convention, or set of rules, that specifies in what order to perform the operations in an algebraic expression. The standard order of operations is as follows: 1. Do the operations in parentheses first. (Use rules 2-4 inside the parentheses.) 2. Calculate all the expressions with exponents. 3. Multiply and divide in order from left to right. 4. Add and subtract in order from left to right. In particular, multiplications and divisions are carried out before additions and subtractions. Example: $\frac{3^2+(6\bullet 2-1)}{5} = \frac{3^2+(12-1)}{5} = \frac{3^2+(11)}{5} = \frac{9+(11)}{5} = \frac{20}{5} = 4$
ordered pair	An <u>ordered pair</u> of numbers is a pair of numbers with a specified order. Ordered pairs are denoted (a, b), (x, y), (s, t), etc. Example: Ordered pairs of numbers are used to represent points in a coordinate plane. The ordered pair (3, -5) represents the point with x-coordinate 3 and y-coordinate -5. This is different from the ordered pair (-5, 3). 
origin	The <u>origin</u> of a coordinate plane is the point (0, 0) where the vertical and horizontal coordinate axes intersect. See <u>coordinate plane</u>.

outcome	An <u>outcome</u> is a result of a probability experiment. Example: If we roll a number cube, there are six possible outcomes: 1, 2, 3, 4, 5, 6.
percent	A <u>percent</u> is a number expressed in terms of the unit $1\% = \frac{1}{100}$. To convert a positive number to a percent, multiply the number by 100. To convert a percent to a number, divide the percent by 100. Example: Fifteen percent $= 15\% = \frac{15}{100} = 0.15$. Example: $4 = 4 \times 100\% = 400\%$.
percent of a number	A <u>percent of a number</u> is the product of the percent and the number. A percent of a number represents the number of parts per 100 parts. Example: 15% of 300 is $\frac{15}{100} \bullet 300 = 45$. Example: If 45 out of 300 students are boys, then 15 out of every 100 students are boys, and 15% of the students are boys.
perimeter	The <u>perimeter</u> of a plane figure is the length of the boundary of the figure. Example: The perimeter of a square is four times its side-length. The perimeter of a rectangle is twice the length plus twice the width. The perimeter of a circular disc is its circumference, which is π times its diameter.
perpendicular	Two lines are <u>perpendicular</u> if they intersect at right angles, that is, they intersect at angles that measure $90°$. Example: 

positive number	A positive number is a number that is greater than zero (written "$a > 0$"). The positive numbers are the numbers to the right of 0 on a number line. Example: The numbers 3, 2.76, and $\frac{3}{7}$ are positive. The numbers -2 and 0 are not positive.
power	See exponential notation.
prime number	A prime number is a natural number that has exactly two factors, namely 1 and itself. Example: The first six prime numbers are 2, 3, 5, 7,11, and 13. Example: 1 is *not* a prime number, because it has exactly one factor.
probability	The probability of an event is a measure of the likelihood of that event occurring. The probability $P(E)$ of the event E occurring satisfies $0 \le P(E) \le 1$. If the event E is certain to occur, then $P(E) = 1$. If the event E is impossible, then $P(E) = 0$. Example: When flipping a coin, the probability that it will land on heads is $\frac{1}{2} = 0.5 = 50\%$.
probability experiment	A probability experiment is an experiment in which the results are subject to chance. Example: Rolling a number cube can be considered a probability experiment.
product	A product is the result of multiplying two or more numbers or expressions. The numbers or expressions being multiplied to form the product are factors of the product. Example: The product of 7 and 8 is 56, written $7 \bullet 8 = 56$. The numbers 7 and 8 are both factors of 56.

quadrant	The coordinate axes of a coordinate plane separate the plane into four regions, called <u>quadrants</u>. The quadrants are labeled $\mathrm{I} - \mathrm{IV}$ starting from the upper right region and going counterclockwise.  Example: The point (8, 5) is located in quadrant I, while (-3, -5) is located in quadrant III.
quotient	In a division problem, the <u>quotient</u> is the result of the division. See <u>division</u>. Example: In $12 \div 3 = 4$, the quotient is 4.
random sample	A <u>random sample</u> is a selection of representatives from a population by chance, in which each member of the population is equally likely to be chosen, and the members of the sample are chosen independently of one another. Example: In order to study how many pets are owned by students at Seaside School, Mr. Goldstein's 7th grade class might survey a random sample of 25 students.
rate	See <u>unit rate</u>.
ratio	A <u>ratio</u> is a pair of nonnegative numbers, not both zero, in a specific order. The ratio of a to b is denoted by $a : b$ (read "a to b," or "a for every b"). Example: The ratio of 3 to 2 is denoted by 3 : 2. The ratio of dogs to cats is 3 to 2. Use 3 cups water for every 2 cups juice.
rational number	A <u>rational number</u> is a number expressible in the form $\frac{m}{n}$, where m and n are integers, and $n \neq 0$. Example: $\frac{3}{5}$ is rational because it is a quotient of integers. Example: $2\frac{1}{3}$ and 0.7 are rational numbers because they can be expressed as quotients of integers, $2\frac{1}{3} = \frac{7}{3}$ and $0.7 = \frac{7}{10}$. Example: $\sqrt{2}$ and π are NOT rational numbers. They cannot be expressed as a quotient of integers.

reciprocal	The <u>reciprocal</u> of a nonzero number is its multiplicative inverse. See <u>division</u>. Example: The reciprocal of 3 is $\frac{1}{3}$. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$.
repeating decimal	A <u>repeating decimal</u> is a decimal that ends in repetitions of the same block of digits. Example: The repeating decimal 52.19343434… ends in repetitions of the block "34." An abbreviated notation for the decimal is $52.19\overline{34}$, where the bar over 34 indicates that the block is repeated. Example: The terminating decimal 4.62 is regarded as a repeating decimal. Its value is 4.620000… .
rounding	<u>Rounding</u> refers to replacing a number by a nearby number that is easier to work with or that better reflects the precision of the data. Typically, rounding requires the changing of digits to zeros. Example: 15,632 rounded to the nearest thousand is 16,000. Example: 12.83 rounded to the nearest tenth is 12.80, or 12.8.
sample	A <u>sample</u> is a subset of the population that is examined in order to make inferences about the entire population. The <u>sample size</u> is the number of elements in the sample. Example: In order to estimate how many radios coming off the production line were defective, the plant manager selected a sample of 12 radios and tested them to see if they worked.
sample space	The <u>sample space</u> for a probability experiment is the set of all possible outcomes of the experiment. Example: In the probability experiment of rolling a number cube, the sample space can be represented as the set {1, 2, 3, 4, 5, 6}.

simplify	Simplify refers to converting a numerical or variable expression to a simpler form. A variable expression might be simplified by combining like terms. A fraction might be simplified by dividing numerator and denominator by a common divisor. Example: $2x + 6 + 5x + 3 = 7x + 9$ Example: $\frac{8}{12} = \frac{2}{3}$
solve an equation	Solve an equation refers to finding all values for the variables in the equation that, when substituted, make the equation true. Values that make an equation true are called solutions to the equation. Example: To solve the equation $2x = 6$, one might think "two times what number is equal to 6?" The only value for x that satisfies this condition is 3, which is the solution.
square number	A square number is a number that is the product of an integer and itself. Example: 9 is a square number, since $3 \bullet 3 = 9$.
subset	A subset is a portion of a set. The set E is a subset of F if every member of E is also a member of F. Example: The set of natural numbers {1, 2, 3, 4,…} is a subset of the set of whole numbers {0, 1, 2, 3, 4,…}.
substitution	Substitution refers to replacing a value or quantity with an equivalent value or quantity. Example: If $x + y = 10$, and we know that $y = 8$, then we may substitute this value for y in the equation to get $x + 8 = 10$.
subtrahend	In a subtraction problem, the subtrahend is the number that is being subtracted from another. See difference. Example: In $12 – 4 = 8$, the subtrahend is 4.

sum	A <u>sum</u> is the result of addition. In an addition problem, the numbers to be added are <u>addends</u>. Example: $\underset{\text{addend}}{7} + \underset{\text{addend}}{5} = \underset{\text{sum}}{12}$ Example: In $3 + 4 + 6 = 13$, the addends are 3, 4, and 6, and the sum is 13.
terminating decimal	A <u>terminating decimal</u> is a decimal whose digits are 0 from some point on. Terminating decimals are regarded as repeating decimals, though the final 0's in the expression for a terminating decimal are usually omitted. See <u>decimal</u>, <u>repeating decimal</u>. Example: 4.62 is a terminating decimal with value $4 + \frac{6}{10} + \frac{2}{100}$.
theoretical probability	The <u>theoretical probability</u> of an event is a measure of the likelihood of the event occurring. Example: In the probability experiment of rolling a (fair) number cube, there are six equally likely outcomes, each with probability $\frac{1}{6}$. Since the event of rolling an even number corresponds to 3 of the outcomes, the theoretical probability of rolling an even number is 3 out of 6, or $3 \bullet \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.
trial	Each performance or repetition of a probability experiment is called a <u>trial</u>. Example: Flipping a coin 25 times can be viewed as 25 trials of the probability experiment of flipping a coin once.
unit fraction	A <u>unit fraction</u> is a fraction of the form $\frac{1}{m}$, where m is a natural number. Example: The unit fractions are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$.
unit rate	The <u>unit rate</u> associated with a ratio $a : b$ of two quantities a and b, $b \neq 0$, is the number $\frac{a}{b}$, to which units may be attached. Example: The ratio of 40 miles each 5 hours has unit rate $\frac{40}{5} = 8$ miles per hour.

value of a ratio	The <u>value of the ratio</u> $a : b$ is the number $\frac{a}{b}$, $b \neq 0$. Example: The value of the ratio $6 : 2$ is $\frac{6}{2} = 3$. Example: The value of the ratio of 3 to 2 is $\frac{3}{2} = 1.5$.
variable	A <u>variable</u> is a quantity whose value has not been specified. Variables are used in many different ways. They may refer to functions, to quantities that vary in a relationship, or to unknown quantities in equations and inequalities. Example: In the equation $d = r\,t$, the quantities d, r, and t are variables. Example: In the equation $6 = 2x + 8$, the variable x may be referred to as the <u>unknown</u>.
variable expression	See <u>expression</u>.
whole numbers	The <u>whole numbers</u> are the natural numbers together with 0. They are the numbers 0, 1, 2, 3, … .
x-axis	The <u>*x*-axis</u> is the horizontal number line passing through the origin in a coordinate plane. See <u>coordinate plane</u>.
y-axis	The <u>*y*-axis</u> is the vertical number line passing through the origin in a coordinate plane. See <u>coordinate plane</u>.

MATHEMATICAL SYMBOLS AND LANGUAGE

Mathematical Symbols			
add	$+$	subtract	$-$
multiply	$\times$ $\bullet$	divide	$\div$ /
is equal to	$=$	is not equal to	$\neq$
is greater than	$>$	is less than	$<$
is greater than or equal to	$\geq$	is less than or equal to	$\leq$
is approximately equal to	$\approx$	parentheses	( )

Symbols for Multiplication

The product of 8 and 4 can be written as:

8 times 4 8×4 $8 \bullet 4$ (8)(4) $\begin{array}{r} 8 \\ \underline{\times\, 4} \end{array}$

The product of 8 and the variable x is written simply as $8x$. We generally avoid using the symbol $\times$ for multiplication because it could be misinterpreted as the variable x. Also, we are cautious about the use of the symbol $\bullet$ for multiplication because it could be misinterpreted as a decimal point.

Symbols for Division

The quotient of 8 and 4 can be written as:

8 divided by 4 $8 \div 4$ $4\overline{)8}$ $\frac{8}{4}$ 8/4

In algebra, the preferred way to show division is with fraction notation.

Meanings for Exponents

In the expression b^n	$b^n = \underbrace{b \bullet b \bullet b \ldots\ldots b \bullet b \bullet b}$
• the number b is the <u>base</u> • the number n is the <u>exponent</u>.	multiplied by itself n times
(base)$^{\text{exponent}}$	$3^4 = 3 \bullet 3 \bullet 3 \bullet 3 = 81$

MATHEMATICAL PROPERTIES

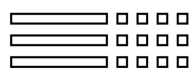
Properties of Arithmetic

<u>Properties of arithmetic</u> govern the manipulation of expressions (mathematical phrases). These include:

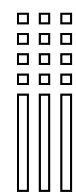
- Associative property of addition
- Commutative property of addition
- Additive identity property
- Additive inverse property
- Associative property of multiplication
- Commutative property of multiplication
- Multiplicative identity property
- Multiplicative inverse property
- Distributive property relating addition and multiplication

Does 14 × 3 Really Have the Same Value as 3 × 14?

The <u>commutative property of multiplication</u> asserts that the product does not depend on the order of the factors. Each of the products 3 × 14 and 14 × 3 is equal to 42.



$$\begin{array}{r} 3 \\ \times\ 14 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 14 \\ \times\ 3 \\ \hline \end{array}$$



Nonetheless, for some problems context is important. Although both actions require 42 marbles, the filling of 3 bags with 14 marbles each will require different supplies than the filling of 14 bags with 3 marbles each.

The Distributive Property

The <u>distributive property</u> relates the operations of multiplication and addition. The term "distributive" arises because the law is used to distribute the factor outside the parentheses over the terms inside the parentheses.

Suppose you earn $9.00 per hour. If you work 3 hours on Saturday and 4 hours on Sunday, one way to compute your earnings is to compute your wages for each day and then add them. Another way is to multiply the hourly wage by the total number of hours. This example illustrates the distributive property.

$$9 \bullet 3 + 9 \bullet 4 = 9(3 + 4)$$

$$27 + 36 = 9(7)$$

THE REAL NUMBER SYSTEM

Subsets of the Real Numbers

The following diagram and explanations outline the real number system. The entire set of real numbers will not be addressed in this course, because a particular subset, irrational numbers, do not arise until 8th grade mathematics. Examples of some irrational numbers are in the diagram below.

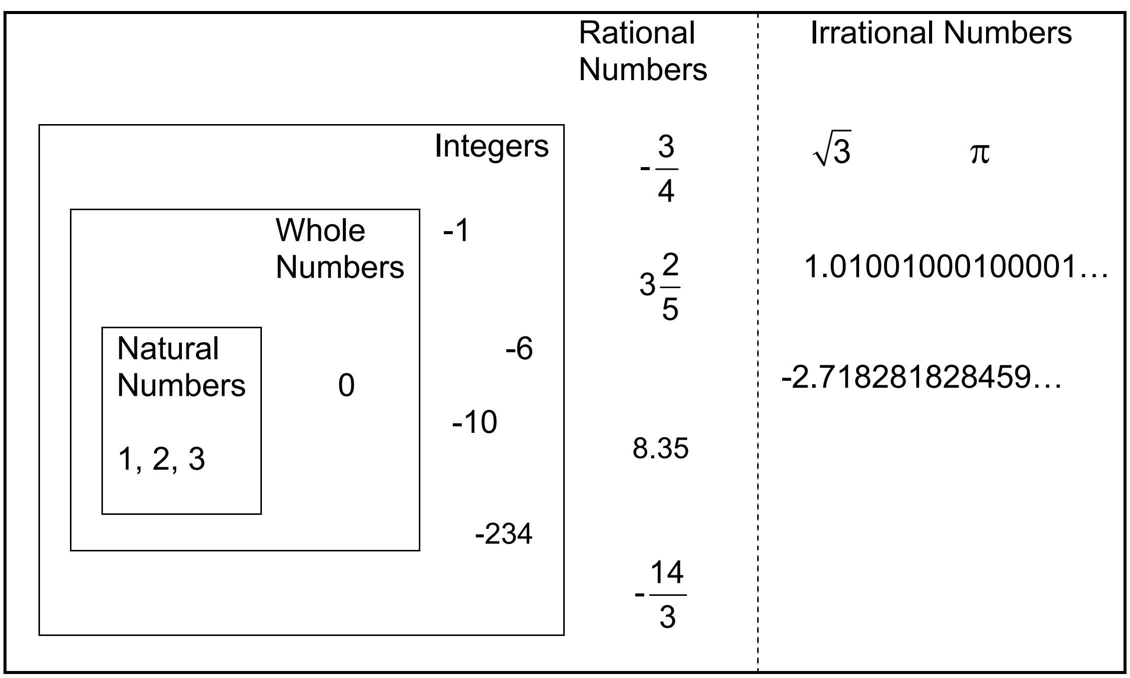
The natural numbers are the numbers 1, 2, 3, … .
Natural numbers are also referred to as counting numbers.

The whole numbers are the natural numbers together with 0.
They are the numbers 0, 1, 2, 3, …

The integers are the whole numbers and their opposites.
They are the numbers 0, 1, 2, 3,….. and -1, -2, -3…

A rational number is a number that can be expressed as a quotient of integers. A number such as $\frac{-3}{4}$ is a quotient of integers. Other numbers, such as $3\frac{2}{5}$ or 11.35 *can be* expressed as quotients of integers (for example, $3\frac{2}{5} = \frac{17}{5}$ and $8.35 = 8\frac{35}{100} = \frac{835}{100}$).

Real Numbers



RATIONAL NUMBER CONCEPTS

Terminating and Repeating Decimals

All rational numbers have decimal expansions that either repeat or terminate.

A repeating decimal is a decimal that ends in repetitions of the same block of digits. For example, the repeating decimal 52.19343434… ends in repetitions of the block "34." An abbreviated notation for the decimal is $52.19\overline{34}$, where the bar over 34 indicates that the block is repeated.

A terminating decimal is a decimal whose digits are 0 from some point on. Terminating decimals are regarded as repeating decimals, though the final 0's in the expression for a terminating decimal are usually omitted. For example, 4.62 is a terminating decimal with the value $4 + 0.6 + 0.02$.

Strategies for Changing Fractions to Decimals

Here are a few sense-making strategies to convert fractions to decimals.

- **Use a visual, such as a hundreds grid**: Fill in 2 out of every 5 squares on a hundreds grid to show that $\frac{2}{5} = \frac{40}{100} = 0.40 = 0.4$.
- **Use a "knowledge of money" strategy**: $\frac{1}{2}$ of a dollar is fifty cents ($0.50), $\frac{1}{4}$ of a dollar is a quarter ($0.25), $\frac{3}{4}$ of a dollar is $0.75, $\frac{1}{10}$ of a dollar is a dime ($0.10), $\frac{1}{20}$ of a dollar is a nickel ($0.05)
- **Use a "doubling" strategy**: $\frac{1}{5} = 0.2 = 0.20$, therefore, $\frac{2}{5} = 2\left(\frac{1}{5}\right) = 2(0.2) = 0.4$.
- **Use a "halving" strategy**: $\frac{1}{4} = 0.25$, therefore $\frac{1}{8} = \frac{1}{2}$ of $\frac{1}{4} = \frac{1}{2}$ of $0.25 = \frac{1}{2}$ of $0.250 = 0.125$.
- **Use a "1 minus a unit fraction" strategy**: $\frac{4}{5} = 1 - \frac{1}{5} = 1 - 0.2 = 0.8$
- **Use a "base-10 denominator" strategy**: $\frac{1}{5} \bullet \frac{2}{2} = \frac{2}{10} = 0.2$ or $\frac{4}{5} = \frac{4}{5} \bullet \frac{2}{2} = \frac{8}{10} = 0.8$
- **Use a "combination" strategy**: $\frac{1}{8} = 0.125$ and $\frac{2}{8} = \frac{1}{4} = 0.25$, so $\frac{3}{8} = \frac{1}{8} + \frac{2}{8} = 0.125 + 0.25 = 0.375$

Use Benchmark Fractions to Write Decimals

We can sometimes use our knowledge of benchmark fractions to find decimal equivalents.

Example 1: Find the decimal representation for $\frac{5}{20}$.

If we know that $\frac{5}{10}$ = 0.5 and that twentieths are half of tenths, we can take half of 0.5, or 0.25, and conclude that $\frac{5}{20}$ = 0.25.

Example 2: Find the decimal representation for $\frac{1}{20}$.

If we know that $\frac{1}{2}$ = 0.5 and that $\frac{1}{20}$ is one-tenth of $\frac{1}{2}$, we can divide 0.5 by 10 and conclude that $\frac{1}{20}$ = 0.05.

Example 3: Find the decimal representation for $\frac{2}{3}$.

If we know that $\frac{1}{3}$ = 0.333… and that $\frac{2}{3}$ is twice $\frac{1}{3}$, then we can double 0.333… and conclude that $\frac{2}{3}$ = 0.666…

Use Decimal Expansion to Change Fractions to Decimals

Example 1: terminating decimal

To show $\frac{7}{8} = 0.875$

$$\frac{7}{8} = (7)\left(\frac{1}{8}\right) = 7(0.125) = 7(0.1) + 7(0.02) + 7(0.005)$$
$$= (0.7) + (0.14) + (0.035)$$
$$= 0.875$$

$$\begin{array}{r} 0.7 \\ 0.14 \\ +\ \underline{0.035} \\ 0.875 \end{array}$$

Example 2: repeating decimal

To show that $\frac{1}{6} = 0.1666...$

$$\frac{1}{6} = \frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{2} \text{ of } 0.33333...$$
$$= \frac{1}{2} \text{ of } 0.3 + \frac{1}{2} \text{ of } 0.03 + \frac{1}{2} \text{ of } 0.003 + ...$$
$$= \frac{1}{2} \text{ of } 0.30 + \frac{1}{2} \text{ of } 0.030 + \frac{1}{2} \text{ of } 0.0030 + ...$$
$$= 0.15 + 0.015 + 0.0015 + ...$$

$$\begin{array}{r} 0.15 \\ 0.015 \\ +\ \underline{0.0015} \\ 0.166\ldots \end{array} \quad \text{(etc.)}$$

Use Division to Change Fractions to Decimals

Example: Change $\frac{3}{8}$ to a decimal.

$$\begin{array}{r} .3\,7\,5 \\ 8\overline{)3.0\,0\,0} \\ -\ \underline{2\,4} \\ 6\,0 \\ -\ \underline{5\,6} \\ 4\,0 \\ -\ \underline{4\,0} \\ 0 \end{array}$$

The decimal representation for $\frac{3}{8}$ is 0.375.

This is a terminating decimal.

Example: Change $\frac{3}{11}$ to a decimal.

$$\begin{array}{r} .2\,7\,2 \\ 11\overline{)3.0\,0\,0\,0} \\ -\ \underline{2\,2} \\ 8\,0 \\ -\ \underline{7\,7} \\ 3\,0 \\ -\ \underline{2\,2} \\ 8\,0 \\ -\ \underline{7\,7} \end{array}$$

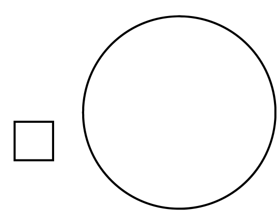
The decimal representation for $\frac{3}{11}$ is $0.27\overline{27}$.

This is a repeating decimal.

Sense-Making Strategies for Comparing and Ordering Fractions

Examples	Name	Ordering Strategy
$\frac{1}{8} < \frac{1}{5} < \frac{1}{4}$	unit fractions	When comparing unit fractions, the fraction with the greater denominator has a smaller value. Think: When you are very hungry, do you want to share a pizza equally with 8 friends or 4? In which situation do you get more pizza?
$\frac{3}{8} < \frac{3}{5} < \frac{3}{4}$	fractions with common numerators	When comparing fractions with common numerators, the fraction with the greater denominator has a smaller value. Using similar reasoning as above: If ONE-fourth is greater than ONE-eighth, then THREE-fourths must be greater than THREE-eighths.
$\frac{1}{12} < \frac{3}{12} < \frac{8}{12}$	fractions with common denominators	When comparing fractions with common denominators, the fraction with the greater numerator has a greater value. Think: A pizza is divided into 8 equal parts. If you eat 1 slice and your friend eats 3 slices, who ate more pizza? THREE-eighths must be greater than ONE-eighth.
$\frac{1}{3} < \frac{1}{2} < \frac{3}{4}$	benchmark fractions	Benchmark fractions are fractions that are easily recognizable, such as $\frac{1}{2}$. For example, $\frac{3}{8} < \frac{1}{2}$, because 3 is less than half of 8.
$\frac{3}{4} < \frac{4}{5} < \frac{7}{8}$	1 minus a unit fraction	All of these are less than 1 whole by a unit fraction. Think about what is missing to make one whole. $\frac{1}{8} + \frac{7}{8} = 1$ and $\frac{1}{4} + \frac{3}{4} = 1$. Since $\frac{1}{4} > \frac{1}{8}$, $\frac{3}{4}$ has a larger missing piece. Therefore $\frac{7}{8} > \frac{3}{4}$.

We assume that all fractions in each example refer to the same whole. This is important because $\frac{9}{10}$ of the area of the square to the right is less than $\frac{1}{2}$ of the area of the circle. In general , $\frac{9}{10} > \frac{1}{2}$ because we assume that we are referring to the same whole.



Fractions, Mixed Numbers, Decimals, and Their Opposites

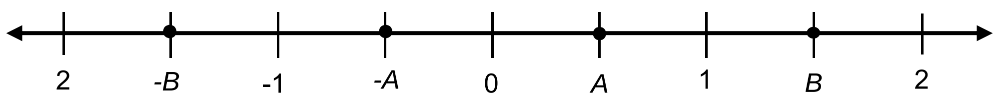
Fractions, mixed numbers, and decimals are all defined as nonnegative numbers. When we name the negative numbers of these forms, we will call them "opposites."

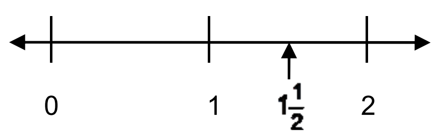
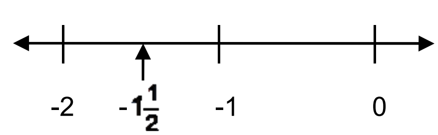
On the number line below, *A* is located at $\frac{1}{2}$ (a fraction) or 0.5 (a decimal).

The opposite of *A* (denoted as -*A*) may be represented by $-\frac{1}{2}$ (opposite of a fraction) or -0.5 (opposite of a decimal). These are both rational numbers.

B is located at $1\frac{1}{2}$ (a mixed number) or 1.5 (a decimal).

The opposite of *B* (denoted as -*B*) may be represented by $-1\frac{1}{2}$ (the opposite of a mixed number) or -1.5 (the opposite of a decimal). These are both rational numbers.



	Mixed Number	Opposite of the Mixed Number
Example	$1\frac{1}{2}$	$-1\frac{1}{2}$
Read as	"one and one-half"	"negative one and one-half" "the opposite of one and one-half"
The sum of	1 and $\frac{1}{2}$	-1 and $-\frac{1}{2}$
Number line representation		
As a fraction (or its opposite)	$\frac{3}{2}$	$-\frac{3}{2}$

INTEGER CONCEPTS

Distance and Absolute Value

The absolute value of a number is its distance from zero on the number line.

A distance 20 units in the positive direction from zero is written $|+20| = 20$.

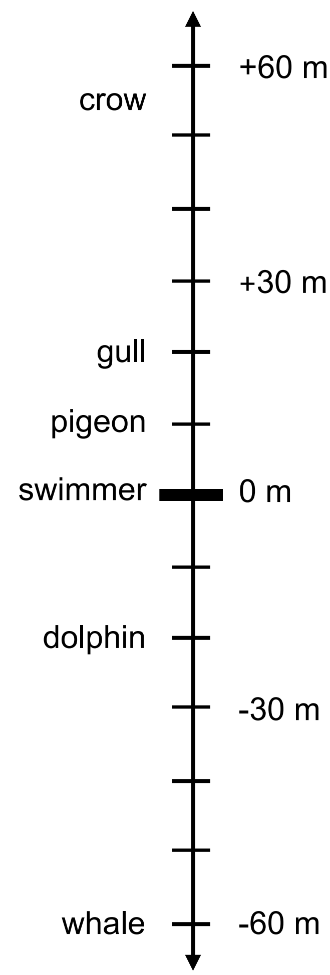
A distance 20 units in the negative direction from zero is written $|-20| = 20$.

Distance is always greater than or equal to zero.

Elevation is a location above, below, or at sea level. Elevation may be positive, negative, or zero.

The vertical number line below represents elevations from 60 meters below sea level (-60 m) to 60 meters above sea level (+60 m).

What	Elevation	Distance from zero (sea level)	Absolute value equation for the distance from sea level
pigeon	+10 m	10 m	$\|10\| = 10$
gull	+20 m	20 m	$\|20\| = 20$
crow	+55 m	55 m	$\|55\| = 55$
swimmer	0 m	0 m	$\|0\| = 0$
dolphin	-20 m	20 m	$\|-20\| = 20$
whale	-60 m	60 m	$\|-60\| = 60$



Here are some true statements about the elevation:
- 10 > 0 (The pigeon is at a higher elevation than the swimmer.)
- 10 > -20 (The pigeon is at a higher elevation than the dolphin.)
- 0 > -20 (The swimmer is at a higher elevation than the dolphin.)

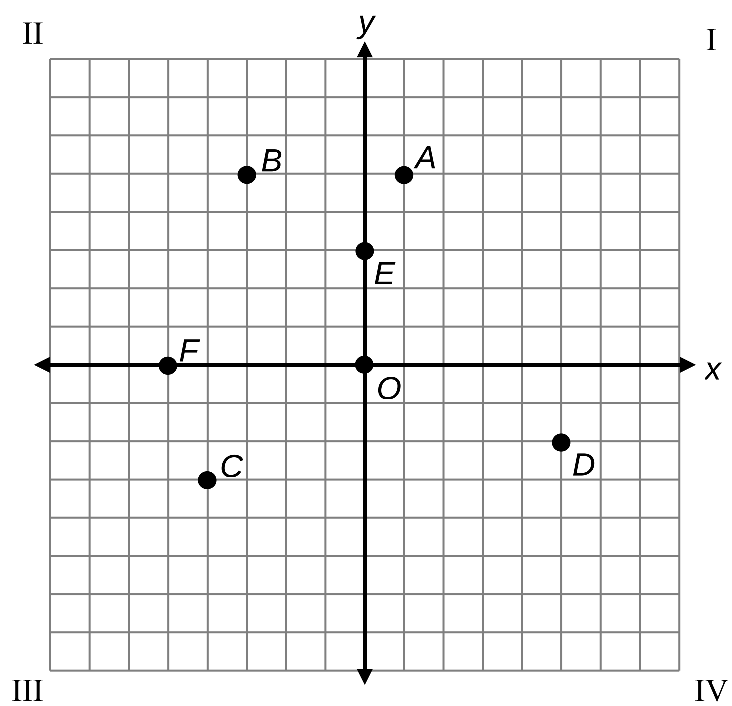
Here are some true statements about the absolute value:
- $|10| > 0$ (The pigeon is farther from sea level than the swimmer.)
- $|-20| > 10$ (The dolphin is father from sea level than the pigeon.)
- $|-20| > 0$ (The dolphin is farther from sea level than the swimmer.)

Graphing in the Coordinate Plane

In the coordinate plane, the horizontal axis is often referred to as the *x*-axis and the vertical axis is referred to as the *y*-axis. These axes divide the plane into four regions, called quadrants. Quadrants are numbered counterclockwise using Roman numerals, beginning in the upper right.

The point *O* (0, 0) is called the origin.

Point	Coordinates	Location
A	(1, 5)	Quadrant I
B	(-3, 5)	Quadrant II
C	(-4, -3)	Quadrant III
D	(5, -2)	Quadrant IV
E	(0, 3)	*y*-axis
F	(-5, 0)	*x*-axis

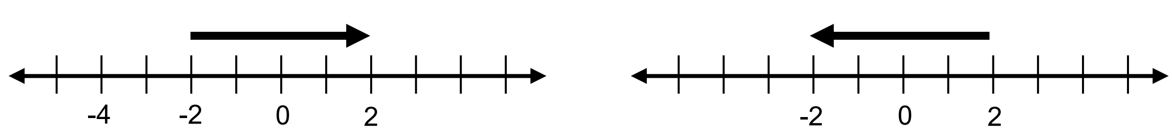


A Linear Model

Numbers may be represented by arrows on a number line. Arrows represent distance (or length) and direction. On a number line, the sign of a number is represented by its direction. The absolute value of a number is represented by the length from its head to its tail.

The first arrow below represents 4. It starts at -2 and ends at 2. Its length is 4.

The second arrow below represents -4. It starts at 2 and ends at -2. Its length is 4.



A Counter Model

The counter model is used to model integers.

Let + represent a positive counter with a value of positive 1

Let – represent a negative counter with a value of negative 1.

A <u>zero pair</u> is a pair with one positive counter and one negative counter.
Both representations below have a value of zero.

one zero pair:	three zero pairs:
+ –	+ + + – – –

Below are some counter diagrams that represent the given integers:

	+4	-2	0
Simplest representation:	+ + + +	– –	(no counters)
Other representations:	+ + + + + –	– – + + – –	+ + – –
	+ + + + + + + – – –	– – – – – – + + + +	+ + + + – – – –

A Temperature Change Model

Suppose scientists discover an amazing way to control the temperature of liquid. They invent "hot pieces" and "cold nuggets" to change the temperature of the liquid. Hot pieces and cold nuggets never melt.

Hot Pieces:

- If you add a hot piece to a liquid, the liquid heats up by one degree.
- If you remove a hot piece from the liquid, the liquid cools down by one degree.

Cold nuggets:

- If you add a cold nugget to the liquid, the liquid cools down by one degree.
- If you remove a cold nugget from the liquid, the liquid heats up by one degree.

How the temperature change model works		**For 1 piece or 1 nugget**
Hot **P**ieces **P**ositive (+)	Put in Heat → Hotter	+ (+1) = 1
	Remove Heat → Colder	- (+1) = -1
Cold **N**uggets **N**egative (−)	Put in Cold → Colder	+ (-1) = -1
	Remove Cold → Hotter	- (-1) = +1

Here are a few examples to show temperature change using hot pieces and cold nuggets

	Simplest ways:		**Other Ways:**	
+4 degrees	Put in 4 hot pieces	Remove 4 cold nuggets	Put in 6 hot pieces and 2 cold nuggets	Remove 6 cold nuggets and 2 hot pieces
-2 degrees	Remove 2 hot pieces	Put in 2 cold nuggets	Remove 3 hot pieces and 1 cold nugget	Put in 3 cold nuggets and 1 hot piece
0 degrees	Do nothing		Put in 4 hot pieces and 4 cold nuggets	Remove 3 hot pieces and 3 cold nuggets

<table>
<tr><th colspan="2">Interpreting the Minus Sign</th></tr>
<tr><td colspan="2">Below are three ways to interpret the minus sign, along with some examples.</td></tr>
<tr><td>Operation Interpretation

When the minus sign is between two expressions, it means “subtract the second expression from the first.”</td><td>Example: $x - 3$

The phrase “x minus 3” can be read:

• x take away 3

• The difference between x and 3

• Subtract 3 from x</td></tr>
<tr><td>Geometric Interpretation

In front of a number or variable, a minus sign means “opposite.” Geometrically, minus can be thought of as a reflection or mirror image. In this case, we are reflecting the number line through zero.</td><td>Example: -3

The phrase “minus 3” can be read:

• Negative 3

Pictorially, this is a location on the number line that is 3 units left of zero.

• Opposite of 3

This is the value you get by first locating 3 on the number line, and then locating that same distance on the opposite side of zero. Geometrically, minus can be thought of as a reflection or mirror Image. In this case, the reflection of 3 through zero is -3.</td></tr>
<tr><td>Algebraic Interpretation

The minus sign is used to show additive inverses. The identity $a + (-a) = 0$ means that $-a$ is the additive inverse of a. It is what we add to a to get 0.</td><td>Example: If $a = -3$, then $-a = 3$

The statement, “If a is equal to minus 3, then minus a is equal to 3” can also be read:

• If a is equal to the opposite of 3, then the opposite of a is equal to 3.

Be careful not to read the variable $-a$ as “negative a” because the variable may not represent a number less than zero.</td></tr>
</table>

INTEGER OPERATIONS

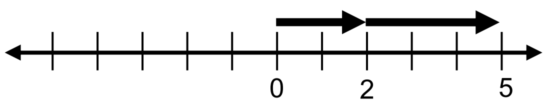
Integer Addition Using Counters		
-3 + (-5) = -8 – – – – – – – – • Start with a work space equal to zero. • Build negative 3. • The (+) means to add. • Add 5 negative counters. • The result is 8 negative counters.	-3 + 5 = 2 – – – + + + + + • Start with a work space equal to zero. • Build negative 3. • The (+) means to add. • Add 5 positive counters. • The result is 2 positive counters.	3 + (-5) = -2 + + + – – – – – • Start with a work space equal to zero. • Build positive 3. • The (+) means to add. • Add 5 negative counters. • The result is 2 negative counters.

Using a Think Aloud Strategy for Addition of Integers	
<u>Think counter model</u>: (begin with a work space equal to zero)	<u>Connect to temperature change model</u>: (begin with temperature equal to zero degrees)
• Build ____________ _____. positive/negative quantity	• Create a liquid temp. of _____ degree(s). number
• The _________ means to <u>add</u>. operation sign	• The __________ means to <u>put in</u>. operation sign
• Add _____ ____________counter(s). quantity positive/negative	• Put in _____ ____________________. quantity hot piece(s) / cold nugget(s)
• The result is _____ _______counter(s). quantity pos. / neg.	• The liquid is now _____ degree(s). number

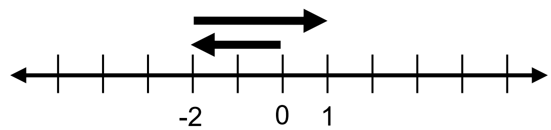
Integer Addition on a Number Line

We can use arrows to represent addition on a number line. The absolute value of a number is represented by the arrow length. The sign of a number is represented by the direction it is pointing. The first arrow begins at zero. The sum is represented by the end (tip) position of the second arrow.

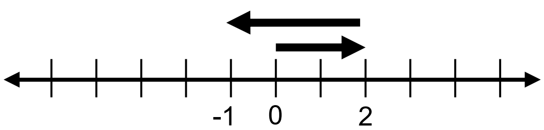
1. (2) + (3) = 5



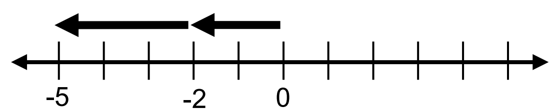
2. (-2) + (3) = 1



3. (2) + (-3) = -1

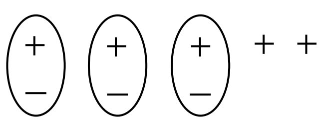


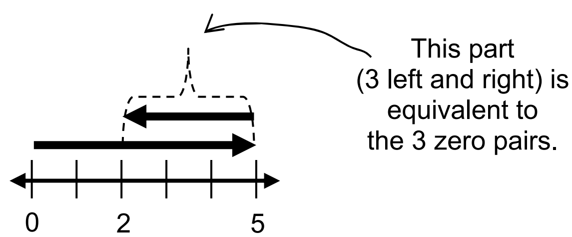
4. (-2) + (-3) = -5



Connecting the Number Line Model and the Counter Model

Below is the example 5 + (-3) = 2 using the counter model and the number line model. Positive counters are represented by an arrow pointing to the right. Negative counters are represented by an arrow pointing to the left. Zero pairs of counters are shown with overlapping arrows.





Rules for Addition of Integers

Rule 1: When the addends have the same sign, add the absolute values. Use the original sign in the answer.

Rule 2: When the addends have different signs, subtract the absolute values. Use the sign of the addend with the greatest absolute value in the answer.

Integer Subtraction Using Counters		
-5 – (-3) = -2 $\cancel{-}\ \cancel{-}\ \cancel{-}\ -\ -$	-3 – (-5) = 2 $\cancel{-}\ \cancel{-}\ \cancel{-}\ \cancel{-}\ \cancel{-}$ $+\ +$	5 – (-3) = 8 $+\ +\ +\ +\ +\ +\ +\ +$ $\cancel{-}\ \cancel{-}\ \cancel{-}$
• Start with a work space equal to zero. • Build negative 5. • The (–) means to <u>subtract</u>. • I do not need zero pairs. • Subtract 3 negative counters. • The result is 2 negative counters.	• Start with a work space equal to zero. • Build negative 3. • The (–) means to <u>subtract</u>. • I need at least 2 zero pairs. • Subtract 5 negative counters. • The result is 2 positive counters.	• Start with a work space equal to zero. • Build positive 5. • The (–) means to <u>subtract</u>. • I need at least 3 zero pairs. • Subtract 3 negative counters. • The result is 8 positive counters.

Using a Think Aloud Strategy for Subtraction of Integers	
<u>Think counter model</u>: (begin with a work space equal to 0)	<u>Connect to temperature change model</u>: (begin with temperature equal to zero degrees)
• Build ________ ______ . positive/negative quantity	• Create a liquid temperature of _____ degree(s). number
• The ________ means to <u>subtract.</u> operation sign	• The __________ means to <u>remove</u>. operation sign
• Add zero pairs if needed.	• Add zero pairs if needed.
• Subtract _____ __________counter(s). quantity positive/negative	• Remove _____ ________________. quantity hot piece(s) / cold nugget(s)
• The result is _____ _______counter(s). quantity pos. / neg.	• The liquid is now _____ degree(s). number

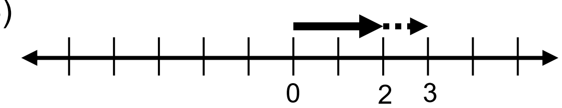
Integer Subtraction on the Number Line

Since addition and subtraction are inverse operations, a subtraction equation may be rewritten as an “adding up” equation. To illustrate subtraction on a number line:

- Rewrite the subtraction equation as its related adding up equation.
- Draw an arrow for the known addend and the missing addend to indicate the sum.
- Record the missing addend (which is also the difference of the subtraction equation.

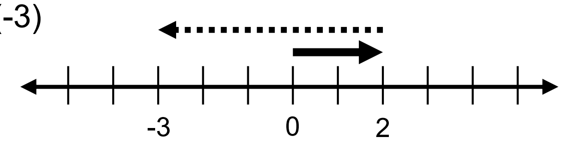
1. (3) – (2) = ___

- Rewrite as ___ + (2) = (3) or (2) + ___ = (3)
- Draw the arrow that represents 2.
- Draw an arrow for the missing addend.
- The missing addend is 1. Therefore, (3) – (2) = 1



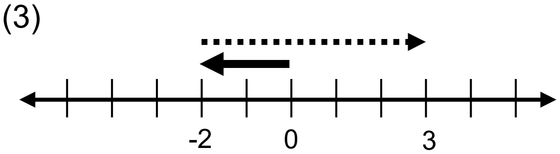
2. (-3) – (2) = ___

- Rewrite as ___ + (2) = (-3) or (2) + ___ = (-3)
- Draw the arrow that represents 2.
- Draw an arrow for the missing addend.
- The missing addend is -5. Therefore, (-3) – (2) = -5



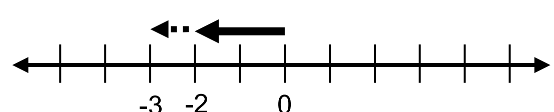
3. (3) – (-2) = ___

- Rewrite as ___ + (-2) = (3) or (-2) + ___ = (3)
- Draw the arrow that represents -2.
- Draw an arrow for the missing addend.
- The missing addend is 5. Therefore, (3) – (-2) = 5



4. (-3) – (-2) = ___

- Rewrite as ___ + (-2) = (-3) or (-2) + ___ = (-3)
- Draw the arrow that represents -2.
- Draw an arrow for the missing addend.
- The missing addend is -1. Therefore, (-3) – (-2) = -1



Rule for Subtraction of Integers

Rule: In symbols, $a - b = a + (-b)$ and $a - (-b) = a + b$.
In words, subtracting a quantity gives the same result as adding its opposite.

Integer Multiplication Using Counters	
2(4) = 8 + + + + + + + + • Start with a work space equal to zero. • The first factor is (+). We will put 2 groups on the workspace. • The second factor is positive. Each group will contain 4 positive counters. • The result is 8 positive counters.	2(-4) = -8 – – – – – – – – • Start with a work space equal to zero. • The first factor is (+). We will put 2 groups on the workspace. • The second factor is negative. Each group will contain 4 negative counters. • The result is 8 negative counters
-2(4) = -8 + + + + – – – – + + + + – – – – • Start with a work space equal to zero. • The first factor is (-). We will remove 2 groups from the workspace. • The second factor is positive. Each group will contain 4 positive counters. • I need at least 8 zero pairs to do this. • The result is 8 negative counters.	-2(-4) = 8 + + + + – – – – + + + + – – – – • Start with a work space equal to zero. • The first factor is (-). We will remove 2 groups from the workspace. • The second factor is negative. Each group will contain 4 negative counters. • I need at least 8 zero pairs to do this. • The result is 8 positive counters.

Think Aloud Strategies for Multiplication of Integers
<u>Think Counter Model</u>: (begin with a work space equal to 0) • The first factor is _________ (positive / negative). We will ______ (place / remove) _____ (quantity) group(s) ________ (in / from) the workspace. • The second factor is _________ (positive / negative). Each group will contain ______ (quantity) _________ (positive / negative) counter(s). • I need at least _____ (quantity) zero pairs to do this. • The result is _____ (quantity) _________ (positive / negative). counter(s).
<u>Connect to the Temperature Change Model:</u> (begin with temperature equal to zero degrees) • The first factor is _________ (positive / negative). We will __________ (put / remove) _____ (quantity) group(s) ______ (in / from) the liquid. • The second factor is _________ (positive / negative). Each group will contain _____ (quantity) ________________ (hot piece(s) / cold nugget(s)). • I need at least _____ (quantity) zero pairs to do this. • The liquid is now _____ (number) degree(s).

Rules for Multiplication of Integers	
Rule 1:	The product of two numbers with the same sign is a positive number.
	Think: (+)(+) = (+) and (-)(-) = (+)
Rule 2:	The product of two numbers with opposite signs is a negative number.
	Think: (+)(-) = (-) and (-)(+) = (-)

Integer Division

Since multiplication and division are inverse operations, many rules for multiplying integers apply to dividing integers as well.

Multiplication Equation	Related Division Equations		
$4 \bullet 5 = 20$	$20 \div 4 = 5$	or	$\frac{20}{4} = 5$
	$20 \div 5 = 4$	or	$\frac{20}{5} = 4$
$4 \bullet (-5) = -20$	$-20 \div 4 = -5$	or	$\frac{-20}{4} = -5$
	$-20 \div (-5) = 4$	or	$\frac{-20}{-5} = 4$
$-4 \bullet (-5) = 20$	$20 \div (-4) = -5$	or	$\frac{20}{-4} = -5$
	$20 \div (-5) = -4$	or	$\frac{20}{-5} = -4$

Rules for Division of Integers

Rule 1: The quotient of two numbers with the same sign is a positive number.

Think: $\frac{(+)}{(+)} = (+)$ and $\frac{(-)}{(-)} = (+)$

Rule 2: The quotient of two numbers with opposite signs is a negative number.

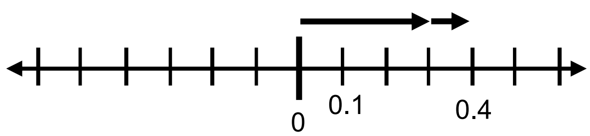
Think: $\frac{(+)}{(-)} = (-)$ and $\frac{(-)}{(+)} = (-)$

RATIONAL NUMBER OPERATIONS

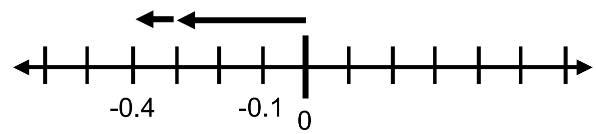
Rational Number Addition on a Number Line

Number lines may be used to extend operations on integers to operations on non-integer rational numbers, like fractions, decimals, and their opposites. Here are some examples that illustrate addition of rational numbers.

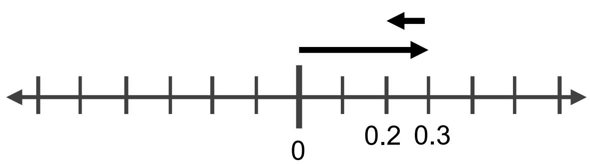
1. $0.3 + 0.1 = 0.4$



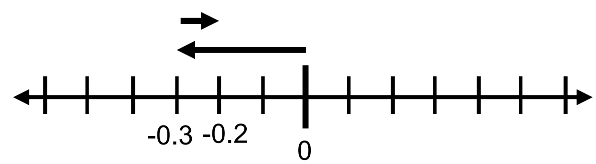
2. $-0.3 + (-0.1) = -0.4$



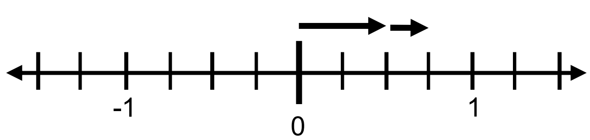
3. $0.3 + (-0.1) = 0.2$



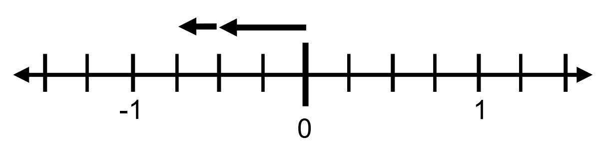
4. $-0.3 + 0.1 = -0.2$



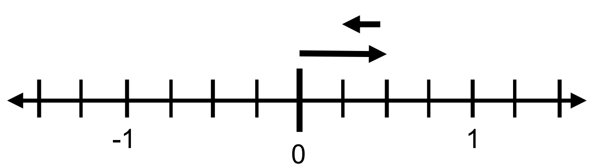
5. $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$



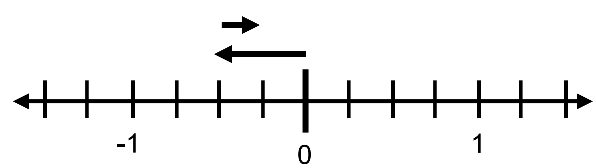
6. $-\frac{1}{2} + \left(-\frac{1}{4}\right) = -\frac{3}{4}$



7. $\frac{1}{2} + \left(-\frac{1}{4}\right) = \frac{1}{4}$



8. $-\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$



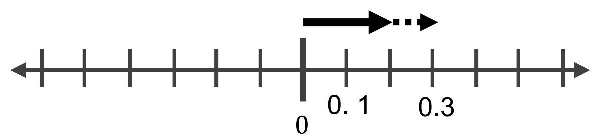
Rational Number Subtraction on the Number Line

Since addition and subtraction are inverse operations, a subtraction equation may be rewritten as an “adding up” equation. To illustrate subtraction on a number line:

- Rewrite the subtraction equation as its related adding up equation.
- Draw an arrow for the known addend and the missing addend to indicate the sum.
- Record the missing addend (which is also the difference of the subtraction equation).

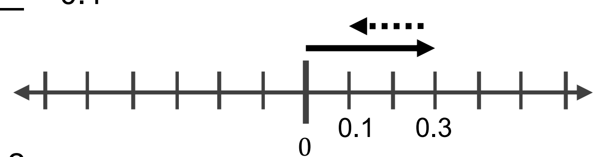
1. 0.3 – 0.2 = ___

- Rewrite as ___ + 0.2 = 0.3 or 0.2 + ___ = 0.3
- Draw the arrow that represents 0.2.
- Draw an arrow for the missing addend.
- The result is 0.1. Therefore, 0.3 – 0.2 = 0.1.



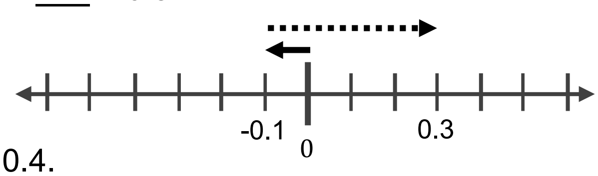
2. 0.1 – 0.3 = ___

- Rewrite as ___ + 0.3 = 0.1 or 0.3 + ___ = 0.1
- Draw the arrow that represents 0.3.
- Draw an arrow for the missing addend.
- The result is -0.2. Therefore, 0.1 – 0.3 = -0.2



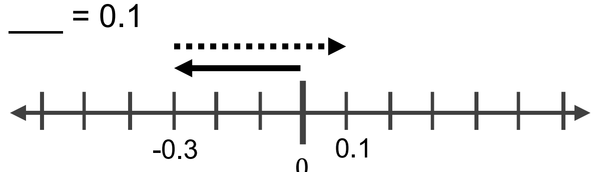
3. 0.3 – (-0.1) = ___

- Rewrite as ___ + (-0.1) = 0.3 or (-0.1) + ___ = 0.3
- Draw the arrow that represents -0.1.
- Draw an arrow for the missing addend.
- The result is 0.4. Therefore, 0.3 – (-0.1) = 0.4.



4. 0.1 – (-0.3) = ___

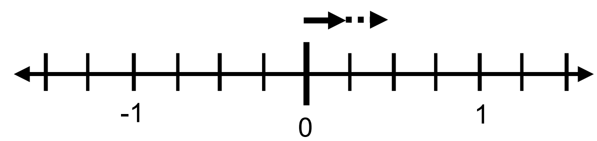
- Rewrite as ___ + (-0.3) = 0.1 or -0.3 + ___ = 0.1
- Draw the arrow that represents -0.3.
- Draw an arrow for the missing addend.
- The result is 0.4. Therefore, 0.1 – (-0.3) = 0.4.



Rational Number Subtraction on the Number Line (Continued)

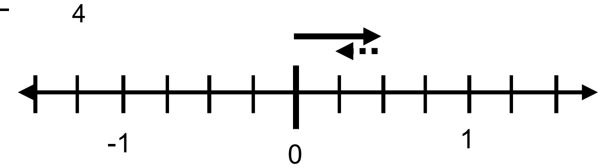
5. $\frac{1}{2} - \frac{1}{4} =$ ____

- Rewrite as ____ $+ \frac{1}{4} = \frac{1}{2}$ or $\frac{1}{4} +$ ____ $= \frac{1}{2}$
- Draw the arrow that represents $\frac{1}{4}$.
- Draw an arrow for the missing addend.
- The result is $\frac{1}{4}$. Therefore, $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.



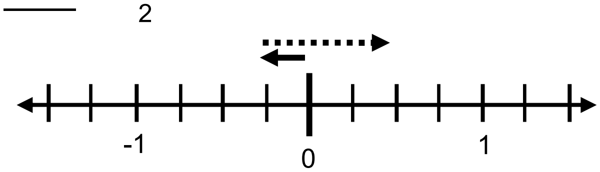
6. $\frac{1}{4} - \frac{1}{2} =$ ____

- Rewrite as ____ $+ \frac{1}{2} = \frac{1}{4}$ or $\frac{1}{2} +$ ____ $= \frac{1}{4}$
- Draw the arrow that represents $\frac{1}{2}$.
- Draw an arrow for the missing addend.
- The result is $-\frac{1}{4}$. Therefore, $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$



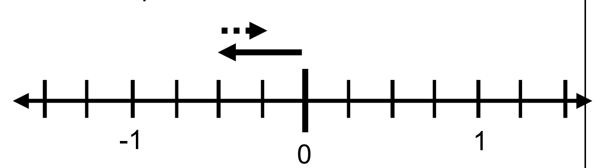
7. $\frac{1}{2} - \left(-\frac{1}{4}\right) =$ ____

- Rewrite as ____ $+ \left(-\frac{1}{4}\right) = \frac{1}{2}$ or $\left(-\frac{1}{4}\right) +$ ____ $= \frac{1}{2}$
- Draw the arrow that represents $\left(-\frac{1}{4}\right)$.
- Draw an arrow for the missing addend.
- The result is $\frac{3}{4}$. Therefore, $\frac{1}{2} - \left(-\frac{1}{4}\right) = \frac{3}{4}$



8. $-\frac{1}{4} - \left(-\frac{1}{2}\right) =$ ____

- Rewrite as ____ $+ \left(-\frac{1}{2}\right) = -\frac{1}{4}$ or $\left(-\frac{1}{2}\right) +$ ____ $= -\frac{1}{4}$
- Draw the arrow that represents $\left(-\frac{1}{2}\right)$.
- Draw an arrow for the missing addend.
- The result is $\frac{1}{4}$. Therefore, $-\frac{1}{4} - \left(-\frac{1}{2}\right) = \frac{1}{4}$



Summary of Rules for Adding and Subtracting Rational Numbers	
Addition	
Rule 1:	When the addends have the same sign, add the absolute values. Use the original sign in the answer.
Rule 2:	When the addends have different signs, subtract the absolute values. Use the sign of the addend with the greatest absolute value in the answer.
Subtraction	
Rule 3:	Subtracting a quantity gives the same result as adding its opposite.

Rational Number Multiplication on a Number Line

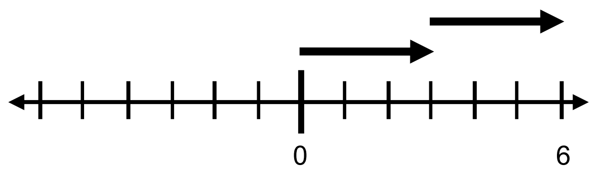
Number lines may used to extend multiplication of integers to non-integer rational numbers, like fractions, decimals, and their opposites. Below are a few examples.

The arrow always begins at zero.

$$(2)(3) = 6$$

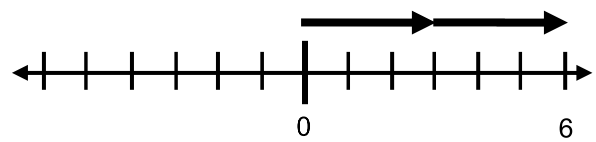
Grouping interpretation

2 arrows of length 3 in the positive direction



Scaling interpretation

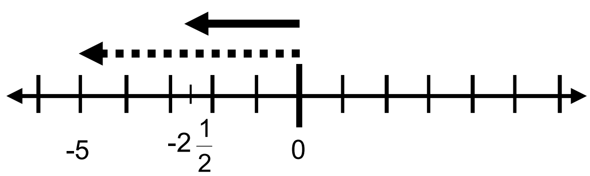
The arrow is stretched twice the length of the arrow from 0 to 3 (view it as one longer arrow).



$$\left(\frac{1}{2}\right)(-5) = -2\frac{1}{2}$$

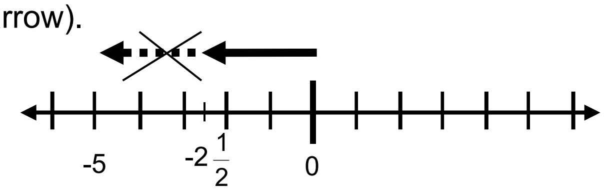
Grouping interpretation

The arrow is $\frac{1}{2}$ the length 5 in the negative direction.



Scaling interpretation

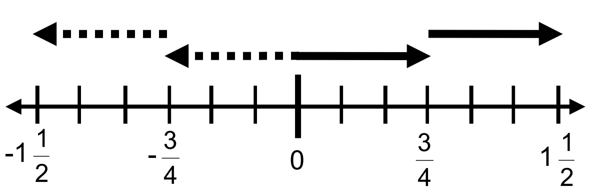
The arrow is shrunk to $\frac{1}{2}$ the length of the arrow from 0 to -5 (view it as one shorter arrow).



$$(-2)\left(-\frac{3}{4}\right) = 1\frac{1}{2}$$

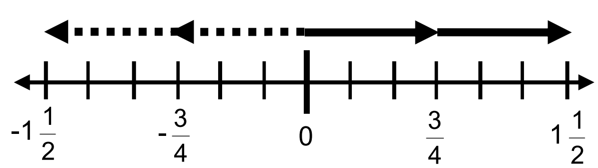
Grouping interpretation

The arrow is the opposite of 2 arrows of length $\frac{3}{4}$ in the negative direction.



Scaling interpretation

The arrow is stretched twice the length in the opposite direction of the arrow from 0 to $-\frac{3}{4}$ (view each as one longer arrow).



Rational Number Division	
Since multiplication and division are inverse operations, the rules for integer operations apply to rational numbers. Below are some examples.	
Multiplication Equation	**Related Division Equations**
$\left(\frac{1}{2}\right)(-5) = -2\frac{1}{2}$	$-2\frac{1}{2} \div (-5) = \frac{1}{2}$ negative ÷ negative = positive $-2\frac{1}{2} \div \frac{1}{2} = -5$ negative ÷ positive = negative

Summary of Rules for Multiplying and Dividing Rational Numbers

Multiplication

Rule 1: The product of two numbers with the same sign is a positive number.

Think: (+)(+) = (+) and (-)(-) = (+)

Rule 2: The product of two numbers with opposite signs is a negative number.

Think: (+)(-) = (-) and (-)(+) = (-)

Division

Rule 1: The quotient of two numbers with the same sign is a positive number.

Think: $\frac{(+)}{(+)} = (+)$ and $\frac{(-)}{(-)} = (+)$

Rule 2: The quotient of two numbers with opposite signs is a negative number.

Think: $\frac{(+)}{(-)} = (-)$ and $\frac{(-)}{(+)} = (-)$

Strategies for Simplifying Complex Fractions

Strategy 1: Write the complex fraction as a division problem.	Strategy 2: Multiply by a form of the "big one" to create a denominator equal to one.
$\dfrac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \bullet \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$	$\dfrac{\frac{1}{2}}{\frac{3}{4}} \bullet \dfrac{\frac{4}{3}}{\frac{4}{3}} = \dfrac{\frac{1}{2} \bullet \frac{4}{3}}{\frac{3}{4} \bullet \frac{4}{3}} = \dfrac{\frac{4}{6}}{1} = \frac{2}{3}$

ORDER OF OPERATIONS

Order of Operations

There are many mathematical conventions that enable us to interpret mathematical notation and to communicate efficiently about common situations. The agreed-upon rules for interpreting mathematical notation, important for simplifying arithmetic and algebraic expressions, are called the <u>order of operations</u>.

1. Do the operations in grouping symbols first (e.g., use rules 2-4 inside parentheses).
2. Calculate all the expressions with exponents.
3. Multiply and divide in order from left to right.
4. Add and subtract in order from left to right.

Example: $$\frac{3^2 + (6 \bullet 2 - 1)}{5} = \frac{3^2 + (12 - 1)}{5} = \frac{3^2 + (11)}{5} = \frac{9 + (11)}{5} = \frac{20}{5} = 4$$

There are many times for which these rules make complete sense and are quite natural. Take this case, for example:

You purchase 2 bottles of water for $1.50 each and 3 bags of peanuts for $1.25 each. Write an expression for this situation, and simplify the expression to find the total cost.

Expression: $$\underbrace{2 \bullet (1.50)} + \underbrace{3 \bullet (1.25)}$$

$$3.00 \quad + \quad 3.75 \quad = \quad \$6.75$$

In this problem, it is natural to find the cost of the 2 bottles of water and then the cost of the 3 bags of peanuts prior to adding these amounts together. In other words, we perform the multiplication operations before the addition operation.

Note however that if we were to perform the operations in order from left to right (as we read the English language from left to right), we would obtain a different result:

$$2(1.50) = 3 \quad \rightarrow \quad 3 + 3 = 6 \quad \rightarrow \quad 6(1.25) = \$7.50$$

Mathematical Separators
Parentheses ( ) and square brackets [ ] are used in mathematical language as separators. The expression inside the parentheses or brackets is considered as a single unit. Algebraic operations are performed inside the parentheses before the expression inside the parentheses is combined with anything outside the parentheses. Example: $5(2 + 1)x = 5(3)x = 15x$ The horizontal line used for a division problem is also a separator. It separates the expressions above and below the line, so the numerator and denominator must be simplified completely before dividing.

Using Order of Operations to Simplify Expressions

Order of Operations	Example $\dfrac{2^3 \div 2(5-2)}{4 + 2 \bullet 10}$	Comments
1. Simplify expressions within grouping symbols.	$\dfrac{2^3 \div 2(3)}{4 + 2 \bullet 10}$	Parentheses are grouping symbols: Therefore $5 - 2 = 3$ The fraction bar, used for division, is also a grouping symbol, so the numerator and denominator must be simplified completely prior to dividing.
2. Calculate powers and roots.	$\dfrac{8 \div 2(3)}{4 + 2 \bullet 10}$	$2^3 = 2 \bullet 2 \bullet 2 = 8.$
3. Perform multiplication and division from left to right.	$\dfrac{12}{4 + 20}$	In the numerator: Divide 8 by 2, then multiply by 3. In the denominator: Multiply 2 by 10.
4. Perform addition and subtraction from left to right.	$\dfrac{12}{24} = \dfrac{1}{2}$	Perform the addition in the denominator: $4 + 20 = 24$ Now the groupings in both the numerator and denominator have been simplified, so the final division can be performed.

RATIOS AND PROPORTIONAL RELATIONSHIPS

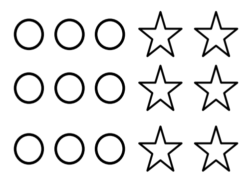
Ratios: Language and Notation

The ratio of a to b is denoted by $a : b$ (read " a to b," or "a for every b"). Here, a and b are nonnegative numbers, at last one of which is nonzero.

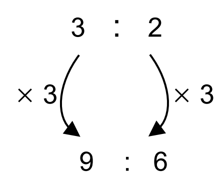
Note that the ratio of a to b is not the same as the ratio of b to a.

We can identify several ratios for the objects in the picture to the right.

- There are 3 circles for every 2 stars.
- The ratio of stars to circles is 2 to 3.
- The ratio of circles to total shapes is 3 : 5
- The ratio of circles to stars is 3 : 2.



If we make 3 copies of the figure above, we get the picture to the right, in which the ratio of circles to stars is 9 : 6. The ratio 9 : 6 is obtained by multiplying each number in the ratio 3 : 2 by 3. In each of the pictures, there are $\frac{3}{2}$ as many circles as there are stars. We refer to $\frac{3}{2}$ as the <u>value</u> of the ratio.



If both numbers in one ratio are multiplied by the same positive number, we arrive at an equivalent ratio. The arrow diagram to the right illustrates that the ratio 3 : 2 is equivalent to the ratio 9 : 6. We will call the number 3 that we multiply by on the sides of the arrow diagram the "multiplier." Equivalent ratios have the same value.

Tables

Tables are useful for recording pairs of numbers that can be represented as ratios. To the right are two ways that numbers might be recorded in a table, based on the pictures above the tables of circles and stars.

Table 1

Circles	3	6	9
Stars	2	4	6

Table 1 is aligned horizontally, and ratios of numbers of circles to stars are pulled from column data. The columns represent equivalent ratios.

Table 2

Circles	**Stars**
3	2
6	4
9	6

Table 2 is aligned vertically, and ratios of numbers of circles to stars are pulled from row data. The rows represent equivalent ratios.

From the tables:

3 : 2, 6 : 4, and 9 : 6 are all equivalent ratios of circles to stars.

2 : 3, 4 : 6, and 6 : 9 are all equivalent ratios of stars to circles.

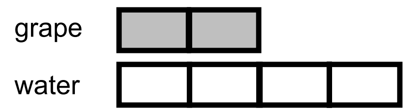
Tape Diagrams

A tape diagram is a visual model consisting of strips divided into rectangular segments whose areas represent sizes of quantities. Tape diagrams are typically used to compare quantities that have the same units.

Here are two versions of tape diagrams that show that the ratio of **grape** juice to **water** in some mixture of grape **drink** is 2 : 4.

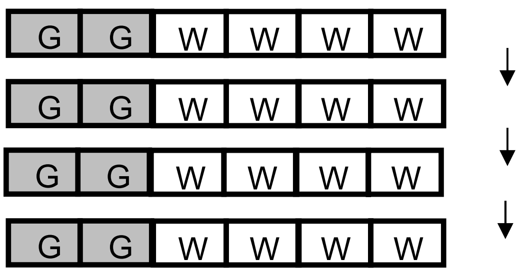
Diagram 1

Diagram 2



Suppose we want to know how much grape juice is needed to make a grape drink mixture that is 24 gallons. We use Diagram 1 for this sort of problem.

Method 1:



2 gal grape : 4 gal water → 6 gal drink (not enough)

4 gal grape : 8 gal water → 12 gal drink (not enough)

6 gal grape : 12 gal water → 18 gal drink (not enough)

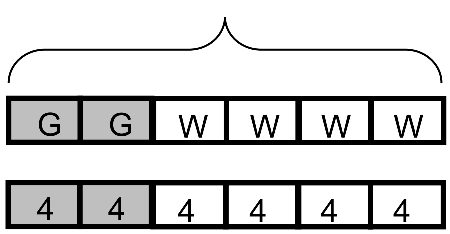
8 gal grape : 16 gal water → 24 gal drink (perfect)

24 gallons of grape drink will require 8 gallons of grape juice.

Method 2:

Since six rectangles in the tape diagram below represent 24 gallons of grape drink, two rectangles represent 8 gallons of grape juice, and each rectangle represents 4 gallons.

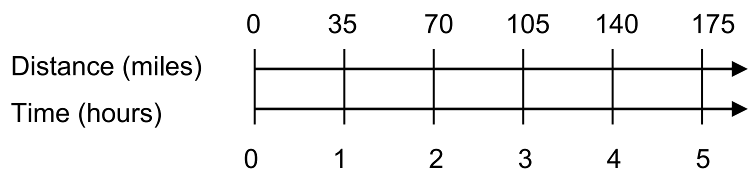
24 gallons



Double Number Line Diagrams

A double number line diagram is a graphical representation of two variables, in which the corresponding values are placed on two parallel number lines for easy comparison. Double number lines are typically used to compare two quantities that have different units.

This double number line shows corresponding ratios for a car that goes 70 miles every 2 hours.



We can see from the double number line diagram that at the given rate, the car drives 35 miles in 1 hour, 105 miles in 3 hours, etc.

Unit Rate and Unit Price

The unit rate associated with a ratio is the value of the ratio, to which units may be attached. In other words, the unit rate associated with the ratio $a : b$ is the number $\frac{a}{b}$, $b \neq 0$. The unit rate usually has units attached, as "something per something" attached.

Example: Suppose a car goes 70 miles every 2 hours.

- This may be represented by the ratio 70 : 2.
- The number $\frac{\text{70 miles}}{\text{2 hours}} = \frac{70}{2}\,\frac{\text{miles}}{\text{hours}} = \frac{35}{1}\,\frac{\text{miles}}{\text{hour}}$ = 35 miles per hour is the unit rate.

A unit price is the price for one unit of a commodity.

Example: Suppose it costs $1.50 for 5 apples.

- This may be represented as the ratio 1.50 : 5.
- The quantity $\frac{\$1.50}{\text{5 apples}}$ = $0.30 per apple is the value of the ratio.
- The unit price could be expressed as a rate in any of the following ways:

0.30 dollars for every one apple	$0.30\,\frac{\text{dollars}}{\text{apple}}$	0.30 dollars per apple	$0.30 per apple

Ratios, Tables, and Graphs

A recipe calls for 2 parts lemon juice for every 3 parts water.

Parts Lemon Juice	2	4	6	8
Parts Water	3	6	9	12

Data from table above can be graphed as ordered pairs in the coordinate plane.

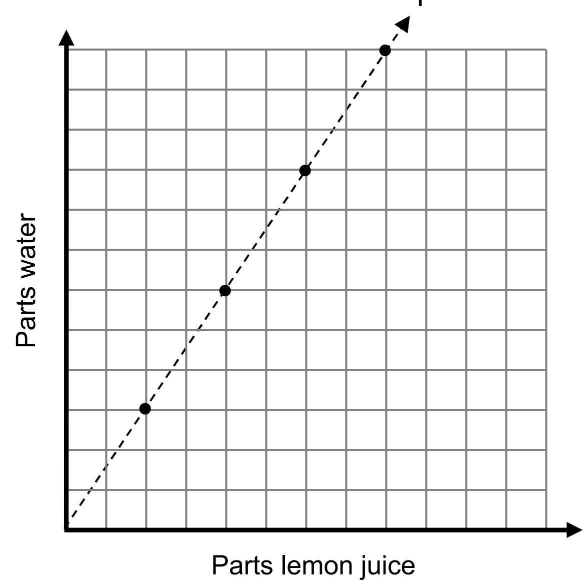
We might choose the variable listed first in the table (parts lemon juice) to be labeled on the horizontal axis, and the variable listed second (parts water) to be labeled on the vertical axis.

Ratios of parts lemon juice to parts water are listed as ordered pairs below. Recall that these are equivalent ratios.

(2, 3) (4, 6)

(6, 9) (8, 12)

Lemon Drink Recipe



The graphed points lie on a straight line that passes through the origin. The part of this line in the first quadrant is a ray that emanates from the origin.

The ray contains the point (1, 1.5). The number 1.5 is the value of the ratio. The unit rate associated with the ratio is 1.5 parts water per 1 part lemon.

PROBABILITY

Phrases That Describe Probabilities

In their assessment reports on climate change, climate scientists attach the following probabilities to common expressions of likelihood:

Virtually certain:	> 99% probability
Extremely likely:	> 95% probability
Very likely:	> 90% probability
Likely:	> 66% probability
More likely than not:	> 50% probability
About as likely as not:	33 to 66% probability
Unlikely:	< 33% probability
Very unlikely:	< 10% probability
Extremely unlikely:	< 5% probability
Exceptionally unlikely:	< 1% probability

Estimating Probabilities from an Experiment

To estimate the probability of an event E, repeat the experiment a number of times and observe how many times the event occurs. The estimate for the probability of the event E occurring is then given by the fraction:

$$\text{estimate} = \frac{\text{number of times an event } E \text{ occurs}}{\text{number of trials}} = \frac{\text{numerator}}{\text{denominator}}$$

In a probability experiment of rolling a (fair) number cube, there are six equally likely outcomes, each with probability $\frac{1}{6}$. The event of rolling an odd number corresponds to three outcomes: 1, 3, or 5. Below is data from an experiment where a cube is rolled 10 times.

Trial #	1	2	3	4	5	6	7	8	9	10
Outcome	4	5	6	3	5	2	1	6	4	2

In this experiment, an odd number occurred 4 times.

$$\text{estimate (odd)} = \frac{4}{10} = \frac{2}{5} = 40\%$$

Since the estimate is based on an experiment, different experiments may lead to different estimates.

Finding Theoretical Probabilities

In probability, the sample space is the set of all possible outcomes. An event is a subset of the sample space. The (theoretical) probability of an event E is the sum of the probabilities of the outcomes that make up E. The probability of the event E is often denoted by $P(E)$.

If the outcomes are equally likely, the probability of an event E is the fraction:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes}} = \frac{\text{success}}{\text{total}}$$

In the probability experiment of rolling a (fair) number cube, there are six equally likely outcomes, each with probability $\frac{1}{6}$. The event of rolling an odd number corresponds to three outcomes: 1, 3, or 5. Thus the theoretical probability of rolling an odd number is given by the fraction:

$$P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2} = 50\%$$

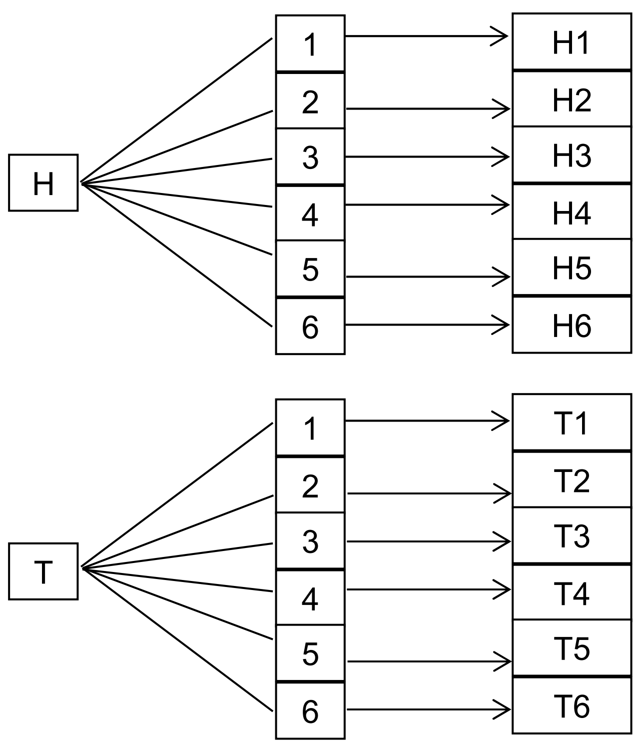
Sample Space Displays

Suppose our experiment is to flip a coin and then roll a number cube. Below are three ways to show all the outcomes (or sample space) of the experiment.

1. Outcome grid:

		Number Roll					
		1	**2**	**3**	**4**	**5**	**6**
Coin Flip	**Heads (H)**	H1	H2	H3	H4	H5	H6
	Tails (T)	T1	T2	T3	T4	T5	T6

2. Tree diagram:



3. List:

H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6

VARIABLES AND EXPRESSIONS

Some Uses of Variables			
Example	**Expression or Equation**	**Variables**	
$4 + 5x$	expression	x	x might be any number; if a value for x is given, the expression can be evaluated.
$12 = 2y + 2$	equation	y	y is an unknown that can be solved for; in this case, y must be 5.
$y = x + 10$	equation	x and y	This equation shows a relation between two variables. As one variable changes, the other changes according to the equation. For example: If $x = 1$, then $y = 11$. If $x = 5$, then $y = 15$.
$y = rx$	equation	x, y, and r	This equation is commonly used to represent a straight line through the origin in the (x,y)-plane. The parameter r may be regarded as a variable, though it is usually fixed for the problem under consideration. When $r > 0$, the variables x and y are in a proportional relationship, and r is the constant of proportionality.
$d = rt$	equation (formula)	d, r, and t	This equation is most commonly used as a formula to relate the quantities of **d**istance, **r**ate of speed, and **t**ime.

Writing Expressions

The specialized system of notation and rules that is used for algebra is sometimes different from the notation used for arithmetic. For example:

- 54 means the sum of five tens and four ones, that is, $5(10) + 4$.
- $5\frac{1}{2}$ means the sum of five and one-half. That is, $5 + \frac{1}{2}$.

However,

- $5x$ means the product of five and x, which can also be written $5(x)$ or $5 \bullet x$. We avoid writing $5 \times x$ because the multiplication symbol $\times$ is easily confused with the variable x.

Equivalent Expressions

Two numerical expressions are <u>equivalent</u> if they are equal.

Example: $2 + 4$ and $-2 + 8$ are equivalent numerical expressions because they are both equal to 6.

Two mathematical expressions are <u>equivalent</u> if for any possible substitution of values for the variables, the two resulting values are equal.

Example: The expressions $x + 2x$ and $4x – x$ are equivalent. For any value of the variable x, the expressions represent the same number. We see this by combining like terms.

$$x + 2x = 3x \quad \text{and} \quad 4x – x = 3x.$$

The expressions x^2 and $2x$ are NOT equivalent. While they happen to be equal if $x = 0$ or $x = 2$, they are not equal for all possible values of x. For instance, if $x = 1$, then $x^2 = 1$ and $2x = 2$.

Properties of arithmetic, such as the distributive property, can be used to write expressions in different, equivalent ways.

Example: $4x + 6x = (4 + 6)x$

Example: $24x + 9x = 3(8x + 3x) = 3x(8 + 3)$

Evaluate or Simplify?

We use the word "evaluate" when we want to calculate the value of an expression for particular values of the variables.

Example: To evaluate $6 + 3x$ when $x = 2$, substitute 2 for x and calculate: $6 + 3(2) = 6 + 6 = 12$.

We use the word "simplify" when we want to rewrite a number or an expression in a form more easily readable or understandable.

Example: To simplify $2x + 3 + 5x$, combine like terms: $2x + 3 + 5x = 7x + 3$.

Sometimes we find the value of a numerical expression by simplifying it.

Example: $16 – 4(2) = 16 – 8 = 8$

Sometimes it may not be clear what the simplest form of an expression is. For instance, by the distributive property, $4(x + 2) = 4x + 8$. For some applications, $4(x + 2)$ may be considered simpler than $4x + 8$, but for other applications, $4x + 8$ may be considered simpler than $4(x + 2)$.

Conjecture and Proof

The differences between inductive reasoning and deductive reasoning, and between conjecture and proof are illustrated by the following exploration of number patterns.

Look at the three sets of consecutive numbers to the right. In each case, the sum of the three consecutive numbers, divided by 3, is the middle number:

2	3	4

11	12	13

66	67	68

$$\frac{2+3+4}{3} = 3 \qquad \frac{11+12+13}{3} = 12 \qquad \frac{66+67+68}{3} = 67$$

Based on this, we might suspect that the sum of any three consecutive numbers, divided by 3, is the middle number. We might check that this is so for several other sums. This might lead us to a conjecture that the sum of any three consecutive numbers, divided by 3 is the middle number.

Above we used inductive reasoning to make a conjecture. But this conjecture has neither been proven to be true nor shown to be false, and it is impossible to check the conjecture one by one on all sets of consecutive numbers. Without algebra, we are stuck.

Let n represent the middle number from a set of three consecutive numbers. Then,

$$\frac{(n-1)+(n)+(n+1)}{3} = \frac{3n}{3} = n\,.$$

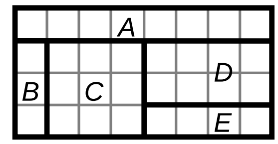
$n-1$	n	$n+1$

This proves the conjecture to be true for ANY number n. We have used deductive reasoning to prove our conjecture. This shows the power of algebra as a tool in mathematics.

Representing Area with Expressions and Equations

Pictured to the right is a large rectangle subdivided into five smaller sub rectangles.

Let A, B, C, D, and E be the areas of the sub rectangles indicated in the figure.



The expression $A + B + C + D + E$ is equal to the combined area of the large rectangle.

Some true equations are:

(1) $A = D$ (2) $\frac{1}{2}D = E$ (3) $B + C = D + E$ (4) $C = 3B$

If we know the value of the variables, we can substitute to find values of other variables.

Suppose that we know that $D = 24$.

Using Equation (1):

$A = D$

$A = 24$

Using Equation (2):

$\frac{1}{2}D = E$

$\frac{1}{2}(24) = E$

$12 = E$

Using Equations (3 and 4):

$B + C = D + E$

$B + 3B = D + E$

$4B = 24 + 12$

$4B = 36$

$B = 9$

EQUATIONS AND INEQUALITIES

Summary of Properties Used for Solving Equations

Properties of arithmetic are used to manipulate expressions (mathematical phrases).

- Associative property of addition
- Commutative property of addition
- Additive identity property
- Additive inverse property
- Associative property of multiplication
- Commutative property of multiplication
- Multiplicative identity property
- Multiplicative inverse property
- Distributive property relating addition and multiplication

Properties of equality are used to manipulate equations (mathematical sentences).

- Addition property of equality
- Subtraction property of equality
- Multiplication property of equality
- Division property of equality

Solving Equations Using Mental Math Strategies

Method 1: To solve an equation using mental math, apply your knowledge of arithmetic facts to find values that make the equation true.

Example 1: Solve $3x = 15$

- Think: ***What number times 3 is 15?***
- Since $3(5) = 15$, $x = 5$.

Example 2: Solve $12 = 20 - k$.

- Think: ***20 minus what equals 12?***
- Since $20 - 8 = 12$, $k = 8$.

Method 2: Use the “cover-up” method:

Example 3: Solve $\frac{n + 20}{3} = 8$

- Cover up $n + 20$ → $\frac{\quad}{3} = 8$
- Think: ***What divided by 3 equals 8?***
- Since $\frac{24}{3} = 8$, you are covering up 24.
- Think: ***What plus 20 equals 24?***
- Since $4 + 20 = 24$, $n = 4$.

Example 4: Solve $-5(m - 2) = -20$.

Cover up $m - 2$ → $-5(\quad) = -20$

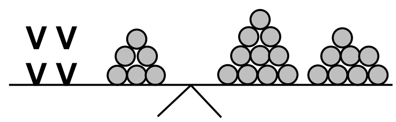
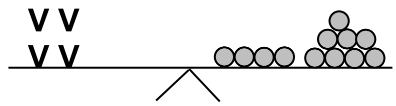
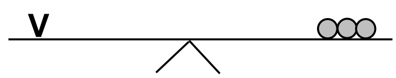
- Think: ***-5 times what equals -20?***
- Since $-5(4) = -20$, you are covering up 4.
- Think: ***What minus 2 equals 4?***
- Since $6 - 2 = 4$, $m = 6$.

Solving Equations Using Balance Strategies

Method 1: Imagine a balance scale as a model.

An equal sign signifies that two expressions have the same value. For our balance scale drawings, **V** represents a cup with an unknown number of marbles in it (the variable *x*), and each cup must have the same number of marbles in it.

The balance scale below illustrates the equation $4x + 6 = 18$. How many marbles are in each cup?

The left side of the scale has 4 cups with the same number of marbles in each plus 6 more marbles		The right side of the scale has 18 marbles.
Now the left side has 4 cups	Remove 6 marbles from each side. 	Now the right side has 12 marbles.
This shows 1 cup	Divide the marbles into 4 equal groups, one for each cup. 	Each cup must contain 3 marbles.

Method 2: Use symbol manipulation.

The example above: $4x + 6 = 18$	**Another example:** $3 = -4 - 2(x - 1)$	**Comments**
	$3 = -4 - 2x + 2$ $3 = -2 - 2x$	Use properties of arithmetic to simplify the expressions on both sides of the equation.
$4x + 6 = 18$ $\underline{-6 \quad -6}$ $4x = 12$	$3 = -2 - 2x$ $\underline{+2} \quad \underline{+2}$ $5 = -2x$	Use the addition property of equality to isolate the variable.
$\frac{4x}{4} = \frac{12}{4}$	$\frac{5}{-2} = \frac{-2x}{-2}$	Use the multiplication property of equality to find the value of the variable (*x*).
$x = 3$	$-\frac{5}{2} = x$	Solution to the equation.
Check: $4(3) + 6 = 18$	Check: $3 = -4 - 2\left(-\frac{5}{2} - 1\right)$	Check using the original equation.

Similar Phrases with Different Meanings

Sometimes it is useful to "translate" a string of words into symbols.

String of Words	Example	Symbols	Classification
is less than	4 is less than 10	$4 < 10$	inequality
less than	4 less than 10	$10 - 4$	expression
is greater than	7 is greater than 2 + 3	$7 > 2 + 3$.	inequality
greater than	7 greater than 2 + 3	$(2 + 3) + 7$.	expression

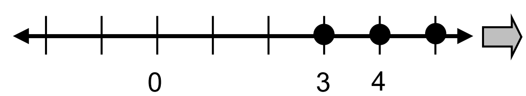
"Is greater than" includes the word "is." Therefore it behaves like a mathematical verb. This string of words is used to make a mathematical sentence (an inequality in this case). "Greater than" is a string of words without a verb. It translates into an expression. In English, we connect phrases with verbs to make sentences. The same is true in mathematics.

Graphing Inequalities

When graphing solutions to inequalities on the number line, we will use arrows to represent sets of solutions that extend indefinitely in one direction or the other. These arrows should not be confused with the arrows used to denote vectors, that is, to denote distance and direction (see Integer Concepts: A Linear Model).

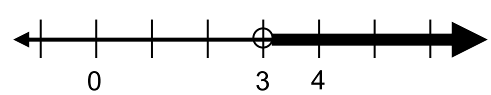
(1) Integers are graphed as dots. We place a large arrow to one side of the number line to indicate that the pattern continues indefinitely.

Example: Integer values that satisfy the inequality $n \geq 3$



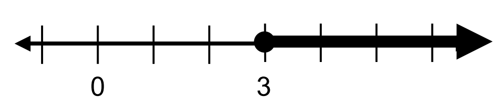
(2) Solutions to inequalities that involve the statements "is less than" (<) and "is greater than" (>) are graphed with an open dot, indicating that the boundary number is not included in the solution set.

Example: All numbers that satisfy the inequality $x > 3$



(3) Solutions to inequalities that involve the statements "is less than or equal to" (≤) and "is greater than or equal to" (≥) are graphed with a solid dot, indicating that the boundary number is included in the solution set.

Example: All numbers that satisfy the inequality $x \geq 3$



Reversing the Direction of an Inequality

When multiplying or dividing both sides of an inequality by a negative number, the direction of the inequality reverses.

Original inequality	Do to both sides	Resulting inequality	Direction reverses?
$10 > -4$	Add 2	$12 > -2$	No
	Subtract 2	$8 > -6$	No
	Multiply by 2	$20 > -8$	No
	Divide by 2	$5 > -2$	No
$-10 < 4$	Add -2	$-12 < 2$	No
	Subtract -2	$-8 < 6$	No
	Multiply by -2	$20 > -8$	Yes
	Divide by -2	$5 > -2$	Yes

The direction of an inequality reverses **ONLY** when multiplying or dividing both sides of an inequality by a negative number.

Note that it does not matter if there are negative numbers in the original inequality or not.

Solving Inequalities in One Variable

When solving a linear inequality, treat the inequality as if it were an equation. When multiplying or dividing both sides of the inequality by a negative number, reverse the direction of the inequality.

Example 1	Comments
$-4x + 1 \le 13$ $-1 \quad\; -1$	(Subtraction) Do not reverse the inequality.
$-4x \le 12$ $\frac{-4x}{-4} \ge \frac{12}{-4}$	(Division by a negative number) **Reverse the inequality symbol.**
$x \ge -3$	Solutions

Example 2	Comments
$2x - 9 \le -13$ $+9 \quad\; +9$	(Addition) Do not reverse the inequality.
$2x \le -4$ $\frac{2x}{2} \le \frac{-4}{2}$	(Division by a positive number) **Do not reverse the inequality symbol.**
$x \le -2$	Solutions

Error alert. The inequality does not always reverse when solving an inequality that includes negatives. Example 2 illustrates this.

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9 7 8 1 6 1 4 4 5 0 7 6 4