# MIDDLE SCHOOL AREA EXPLORATIONS

Presented by

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Excerpts from *MathLinks*: Grades 6 and 7

# STANDARDS FOR MATHEMATICAL CONTENT

### OVERVIEW

The Standards for Mathematical Content outline the skills and understandings students will learn by grade level or course, and are organized around domains and coherent clusters.

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<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td></tr>
<tr><td colspan="6">Numbers and Operations in Base Ten</td><td colspan="3">The Number System</td></tr>
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# THE CCSS SHIFTS

**Seeking increased FOCUS, COHERENCE, and RIGOR**

| **Focus** | **Coherence** | **Rigor** |
|---|---|---|
| • Strongly emphasized at appropriate grade levels for deep understanding.<br><br>• Fewer topics and standards per grade level allow for greater depth and focus; avoids the rush to "cover" content, and is counter to the "mile-wide-inch-deep" approach. | • Thinking across grade levels, and linking to major topics within grade levels.<br><br>• Important ideas can be linked across any one grade level (horizontally), and also from one grade level to the next (vertically). | • Pursue, with equal intensity in major topics:<br>(1) conceptual understanding;<br>(2) procedural skill and fluency;<br>(3) problem solving and application.<br><br>• The recent culture in mathematics education was to concentrate mainly on procedural skill and fluency (think standardized testing). |

# COMMON CORE STATE STANDARDS – MATHEMATICS

## STANDARDS FOR MATHEMATICAL CONTENT

| | |
|---|---|
| **6.EE.A** | **Apply and extend previous understandings of arithmetic to algebraic expressions.** |
| 6.EE.2a | Write, read, and evaluate expressions in which letters stand for numbers: Write expressions that record operations with numbers and with letters standing for numbers. For example, express the calculation "Subtract y from 5" as 5 – y. |
| 6.EE.2c | Write, read, and evaluate expressions in which letters stand for numbers: Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). *For example, use the formulas $V = s^3$ and $A = 6\ s^2$ to find the volume and surface area of a cube with sides of length s = 1/2.* |
| 6.EE.4 | Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.* |
| **6.G.A** | **Solve real-world and mathematical problems involving area, surface area, and volume.** |
| 6.G.1 | Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. |
| **7.NS.A** | **Apply and extend previous understandings of multiply, and divide rational numbers.** |
| 7.NS.3 | Solve real-world and mathematical problems involving the four operations with rational numbers. |
| **7.EE.A** | **Use properties of operations to generate equivalent expressions.** |
| 7.EE.2 | Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."* |
| **7.G.B** | **Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.** |
| 7.G.4 | Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. |
| 7.G.6 | Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. |

# STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe a variety of processes and proficiencies that educators seek to develop in mathematically proficient and fluent students across all grade levels.

Many processes and proficiencies in these practice standards overlap, several may be used together on any given problem or task, and rarely would we expect students to use them all at once. We do expect that over time students will use them frequently. In addition, some will be used naturally within the context of solving particular problems, and others will only occur in an environment in which students are provided ample opportunities to collaborate and discuss.

One way to think about the practices is in groupings (graphic from CCSS-M author, Bill McCallum).

**HABITS OF MIND**

MP1 Make sense of problems and persevere in solving them
MP6 Attend to precision

**REASONING AND EXPLAINING**

MP2 Reason abstractly and quantitatively
MP3 Construct viable arguments and critique the reasoning of others

**MODELING AND USING TOOLS**

MP4 Model with mathematics
MP5 Use appropriate tools strategically

**SEEING STRUCTURE AND GENERALIZING**

MP7 Look for and make use of structure
MP8 Look for and make use of repeated reasoning

From CCSS-M:

“Students who lack understanding of a topic may rely on procedures too heavily...In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. Without understanding, a student may rely on procedures and may not represent problems coherently, justify conclusions, apply the mathematics to other situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an interview, or deviate from a known procedure to find a shortcut.”

# STANDARDS FOR MATHEMATICAL PRACTICE (Continued)

| Habits of Mind | |
|---|---|
| **MP1**<br>**Make sense of problems and persevere in solving them** | **MP6**<br>**Attend to precision** |
| • Understand a problem and look for entry points<br>• Consider given information, constraints, and relationships<br>• Consider  simpler or analogous problems<br>• Make conjectures, monitor progress and alter their solution course as needed<br>• Explain correspondences using multiple representations<br>• Understand and analyze the approaches of others<br>• Continually ask, “Does this make sense?” | • Communicate mathematical ideas precisely<br>• Use clear definitions<br>• State meaning of symbols and use them properly<br>• Attend to units of measures and labeling of axes<br>• Calculate accurately and give solution with appropriate degree of accuracy |

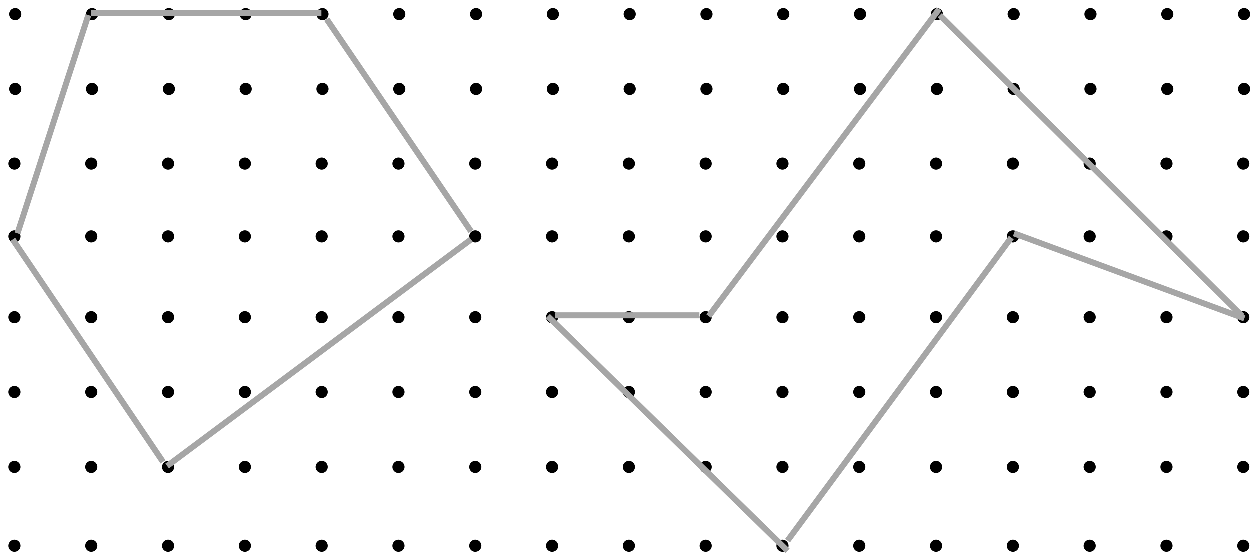
| Reasoning and Explaining | |
|---|---|
| **MP2**<br>**Reason abstractly and quantitatively** | **MP3**<br>**Construct viable arguments and critique the reasoning of others** |
| • Attend to the meaning of quantities<br>• Decontextualize a problem using symbols, and manipulate them as if they have a life of their own<br>• Contextualize manipulations to create a coherent representation of a problem | • Use assumptions, definitions, and established results to create arguments (deductive reasoning)<br>• Make and test conjectures based on evidence (inductive reasoning)<br>• Analyze situations by breaking them into cases<br>• Use counterexamples to disprove a statement<br>• Identify flaws in an argument<br>• Listen to or read to arguments and ask useful questions to clarify reasoning |

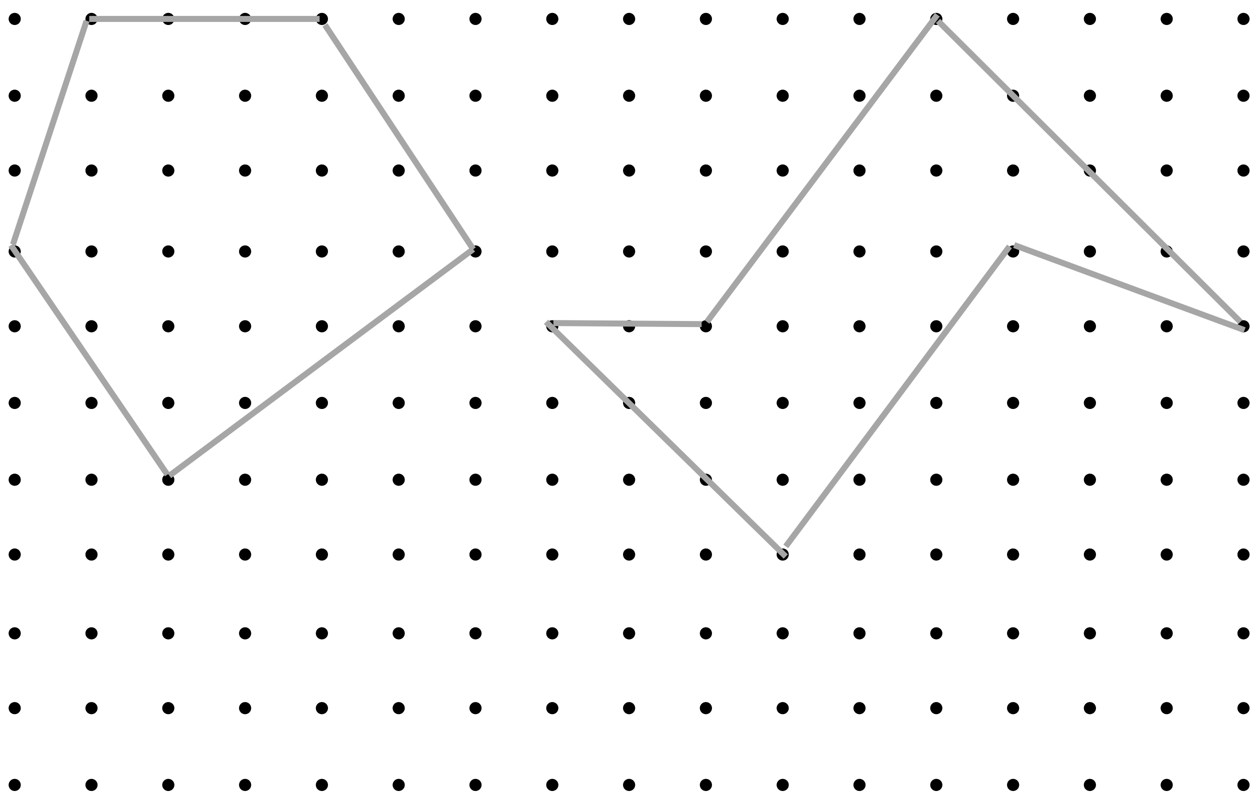
# STANDARDS FOR MATHEMATICAL PRACTICE (Continued)

| Modeling and Using Tools | |
|---|---|
| **MP4**<br>**Model with Mathematics** | **MP5**<br>**Use appropriate tools strategically** |
| • Apply mathematics to solve everyday problems<br>• Make reasonable assumptions and approximations to simplify a situation<br>• Identify important quantities in a situation<br>• Use multiple representations to analyze relationships and draw conclusions<br>• Interpret results in the context of the situation<br>• Improve the mathematical approach (model) if it has not served its purpose | • Select useful tools such as paper and pencil, graph paper ruler, calculator, concrete model, spreadsheet, or statistical software to solve problems<br>• Use concrete models and technology tools to explore concepts<br>• Recognize limitations of tools<br>• Identify and use relevant external resources, such as the internet |

| Structure and Generalizing | |
|---|---|
| **MP7**<br>**Look for and make use of structure** | **MP8**<br>**Look for and make use of repeated reasoning** |
| • Identify patterns and apply them to solve problems<br>• Recognize the structure of a symbolic representation and generalize it<br>• See complicated objects as composed of chunks of simpler objects | • Identify repeated calculations and patterns<br>• Generalize procedures based on repeated patterns or calculations<br>• Find shortcuts based on repeated patterns or calculations |

# COUNTING SQUARES





# AREAS OF POLYGONS

**Summary**

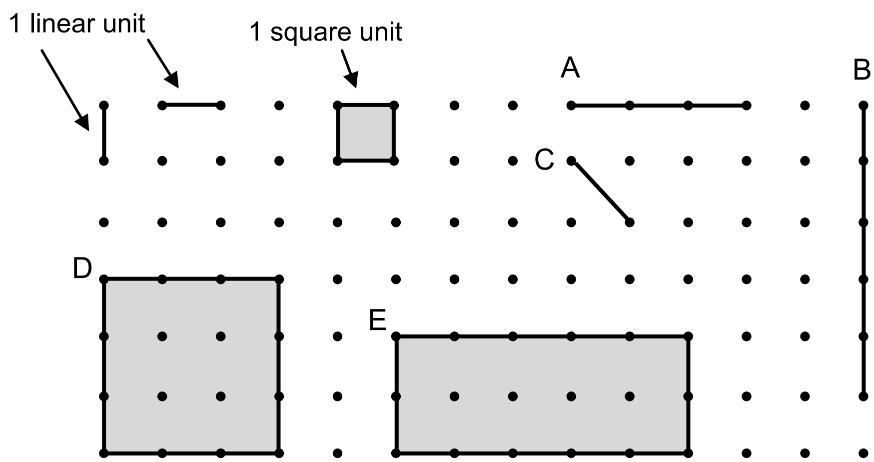
We will establish formulas for areas of parallelograms, triangles, and trapezoids. We will find areas of irregular polygons. We will apply area formulas to solve problems.

**Goals**

- Derive formulas for the areas of parallelograms, triangles, and trapezoids.
- Find areas of irregular polygons.
- Solve real-world and mathematical problems that involve area.

**Warmup**

One unit of length (1 linear unit) and one unit of area (1 square unit) are defined below.



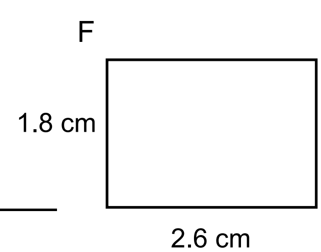

1. Find the length of the segment labeled: A. __________ B. __________

2. Is the length of the segment labeled C equal to 1 linear unit? __________
   Explain how you know.

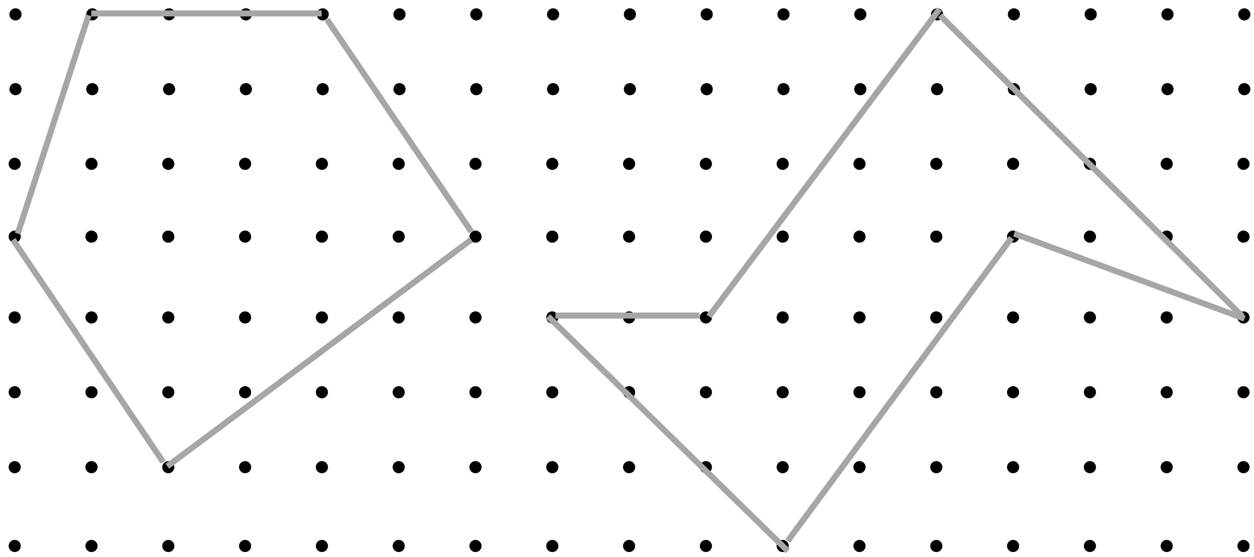
3. Find the area of the figure labeled D. __________

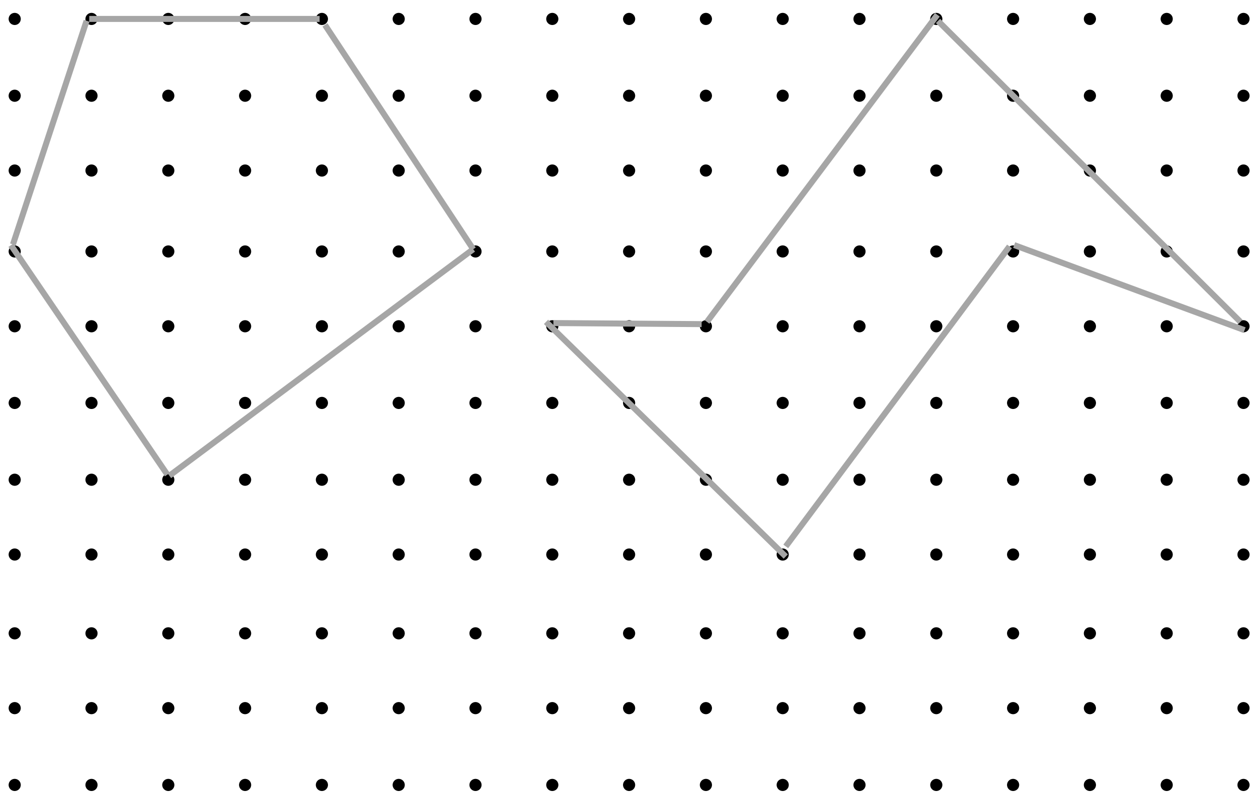
4. Find the area of the figure labeled E. __________

5. Find the area of the rectangle to the right labeled F. __________



# MAKING RECTANGLES



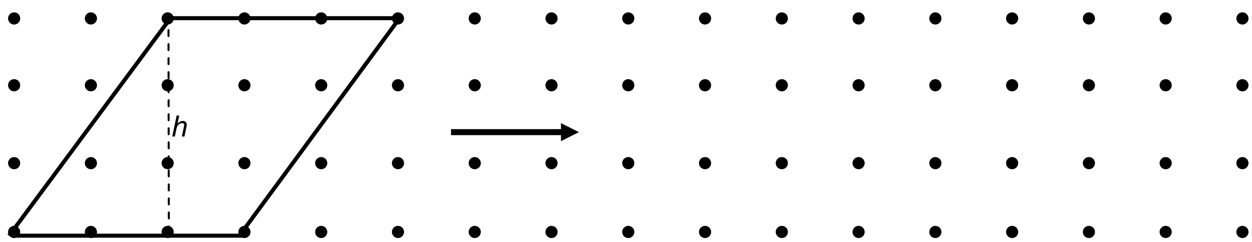


# AREAS OF PARALLELOGRAMS

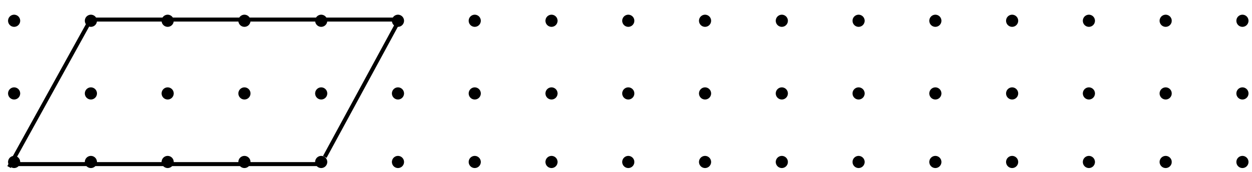
A <u>parallelogram</u> is a quadrilateral in which opposite sides are parallel. In a parallelogram, opposite sides have equal length, and opposite angles have equal measure.

- Use a straightedge to draw an altitude of each parallelogram.
- Rearrange the pieces of the parallelogram to form a rectangle with the same area.
- Draw the rectangle next to the corresponding parallelogram.
- Find the area of each figure.

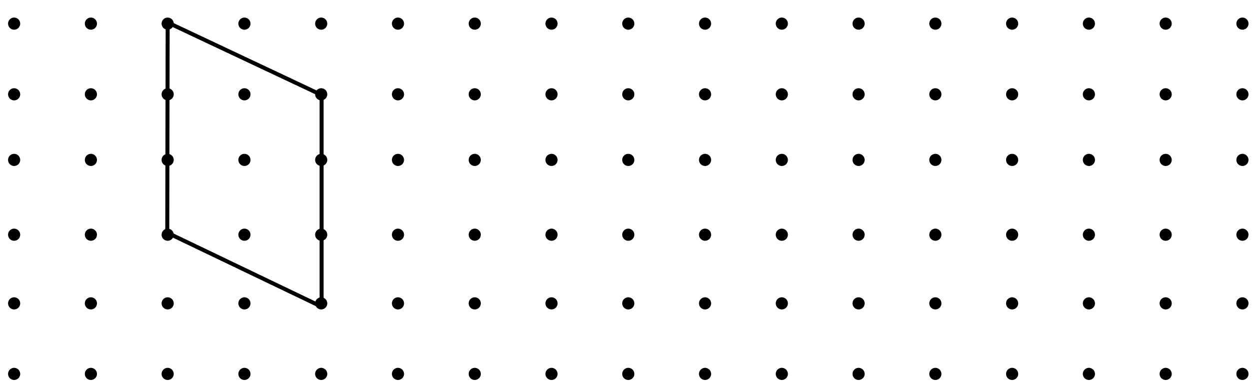
1.




2.



3.



4. Is a rectangle also a parallelogram? Explain.

5. Write in words: The area of a parallelogram is equal to

6. Write in symbols: $A_{\text{▱}}$ = ____________________

# AREAS OF TRIANGLES

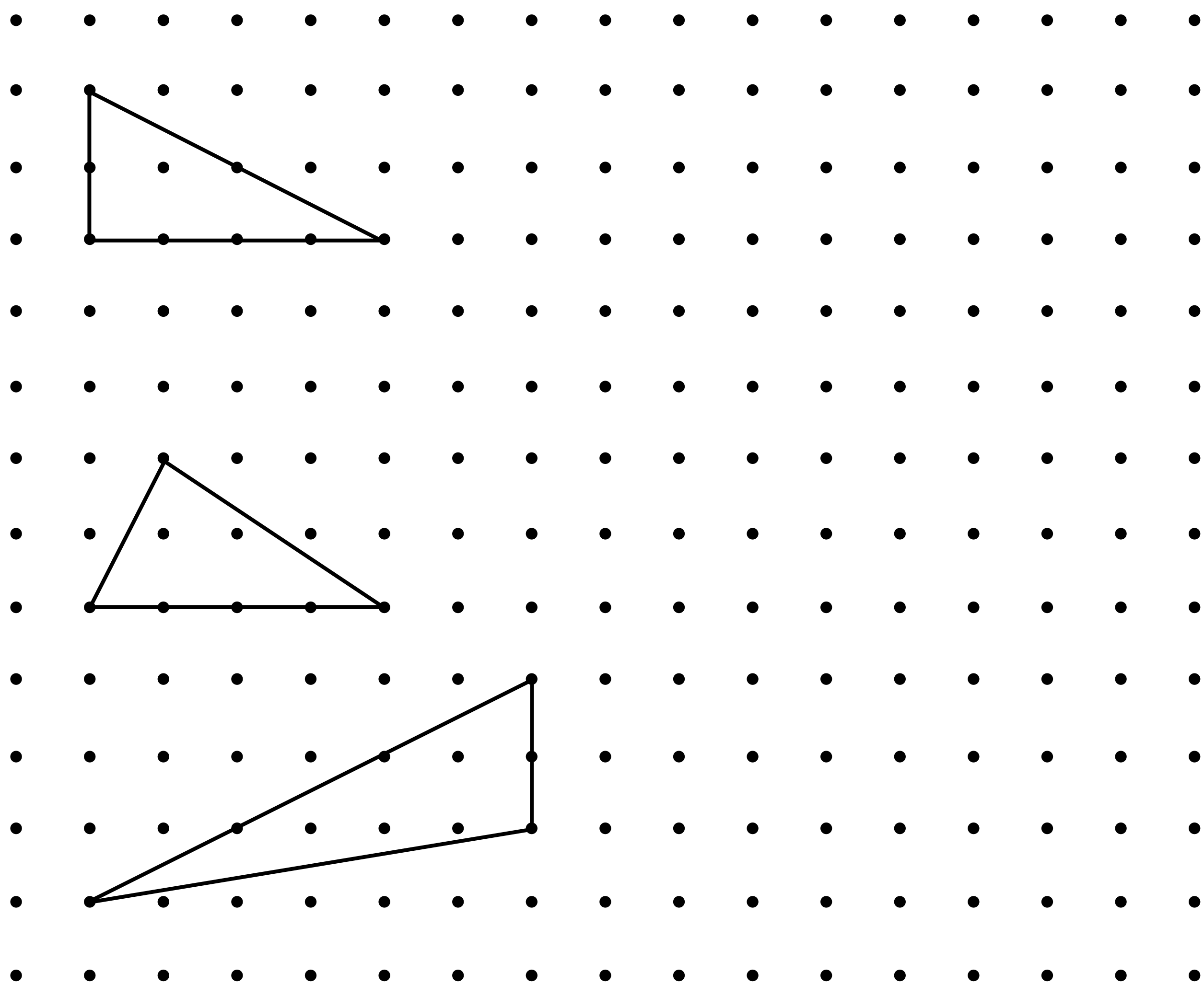
A triangle is a polygon with three sides.

- Duplicate (or copy) each triangle.
- Rearrange the two triangular pieces to form a rectangle or a parallelogram.
- Draw the rectangle or parallelogram next to the corresponding triangle or on the triangle.
- Find the area of each triangle.

1.

2.

3.



4. Write in words: The area of a triangle is equal to

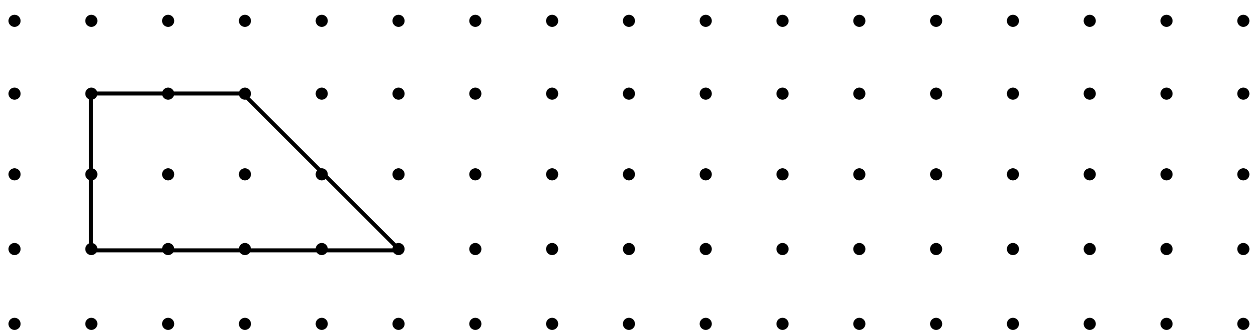
5. Write in symbols: $A_{\triangle}$ = ________________

# AREAS OF TRAPEZOIDS

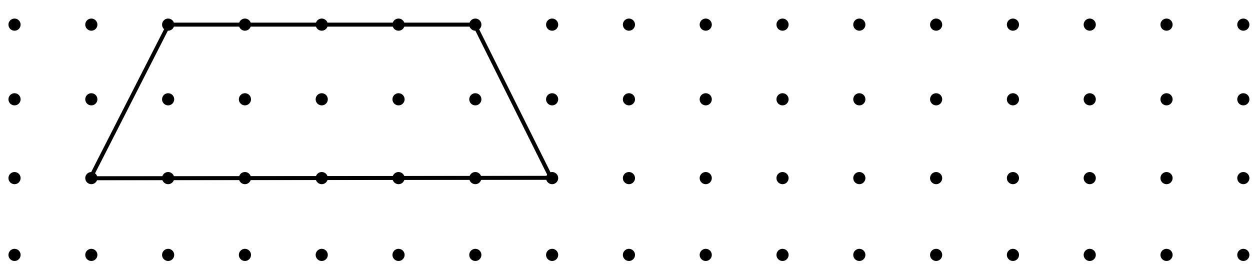
A <u>trapezoid</u> is a quadrilateral which has at least one pair of parallel sides.

- Duplicate each trapezoid.
- Rearrange the two pieces to form a rectangle or a parallelogram.
- Draw the rectangle or parallelogram next to the corresponding trapezoid or on the trapezoid.
- Find the area of each trapezoid.

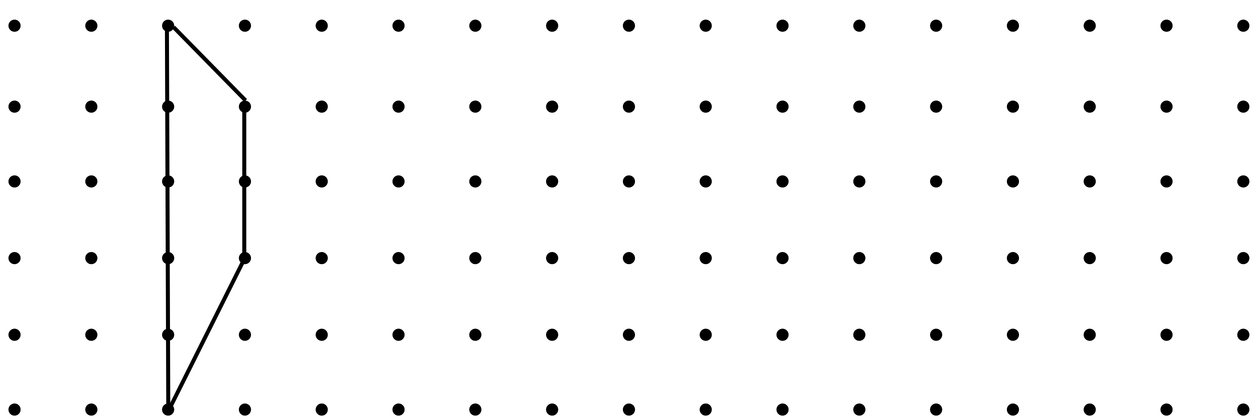
1.



2.



3.



4. Write in words: The area of a trapezoid is equal to

5. Write in symbols: *A*⏢ = ________________

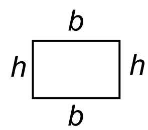
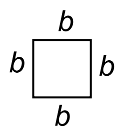
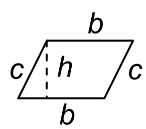
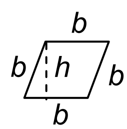
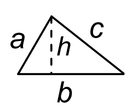
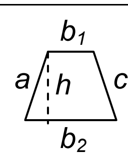
# AREA FORMULAS

Fill in the table below when instructed by your teacher.

| Shape with<br>Definition or Description | Sketch | Area Formula |
|---|---|---|
| 1. Rectangle: | | |
| 2. Square: | | |
| 3. Parallelogram: | | |
| 4. Rhombus: | | |
| 5. Triangle: | | |
| 6. Trapezoid: | | |

# TEACHER CONTENT INFORMATION

## MATH NOTES

| Summary of Perimeter and Area Formulas | | | |
|---|---|---|---|
| **Shape/Definition** | **Diagram** | **Perimeter** | **Area** |
| **Rectangle**<br>a quadrilateral with 4 right angles |  | $P = 2(b + h)$<br>or<br>$P = 2b + 2h$ | $A = bh$ |
| **Square**<br>a rectangle with 4 equal side lengths |  | $P = 4b$ | $A = b^2$ |
| **Parallelogram**<br>a quadrilateral with opposite sides parallel |  | $P = 2(b + c)$<br>or<br>$P = 2b + 2c$ | $A = bh$ |
| **Rhombus**<br>a quadrilateral with 4 equal side lengths |  | $P = 4b$ | $A = bh$ |
| **Triangle**<br>a polygon with three sides |  | $P = a + b + c$ | $A = \frac{1}{2}bh$ |
| **Trapezoid**<br>A quadrilateral with a pair of parallel sides |  | $P = a + b_1 + b_2 + c$ | $A = \frac{1}{2}(b_1 + b_2)h$ |

To make the relationships among formulas more explicit, the base of each figure is denoted as $b$. This means that some formulas will not be expressed using conventional notation. For example, the perimeter of a square is typically written as $P = 4s$, where $s$ = the length of the side of the square.

### Different Definitions for Quadrilaterals

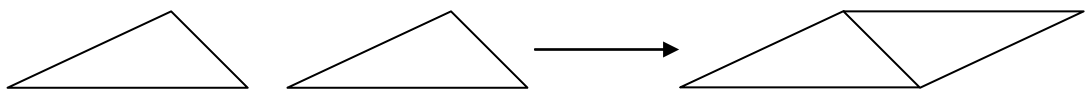
Is a rectangle a trapezoid? Is a square a rhombus? To a certain extent, the answers to these questions are a matter of convention and historical whim. According to the definitions we have given, a parallelogram is a trapezoid, and a square is a rhombus. However, there is room for disagreement. Some textbook writers do not allow parallelograms to be trapezoids, and some textbook writers do not allow squares to be rhombi.

To complicate the issue further, definitions sometimes vary from one country to another. In England a “trapezium” is a trapezoid, while in the United States a “trapezium” is a quadrilateral in which no two sides are parallel.

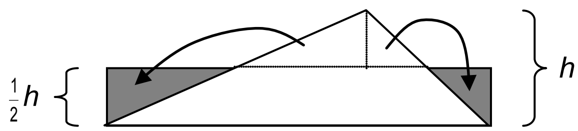
## MATH NOTES (Continued)

### The Area of a Triangle by Duplication and Dissection

Method #1: Duplicate the triangle (create a congruent triangle), and reassemble the two congruent triangles as indicated to form a parallelogram. Since congruent triangles have equal areas, the area of the triangle is half the area of the parallelogram.



Method #2: Orient triangle with the longest side as the base. Cut the triangle along a line parallel to the base and at half the height. Cut the top triangular piece along the perpendicular to the base from the top vertex. The two resulting pieces from the top can be positioned over the shaded pieces of the rectangle with the same base as the triangle and half the height, so that the three pieces from the triangle cover the rectangle exactly once.



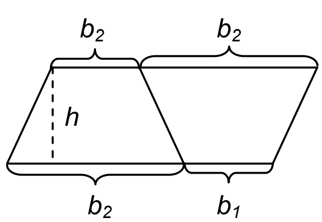

### The Area of a Trapezoid by Duplication and Dissection

Method #1: Duplicate the trapezoid (create a congruent trapezoid), and reassemble the two congruent trapezoids as indicated to form a parallelogram. The parallelogram will have the same height as the trapezoid, and the base of the parallelogram will be equal to the sum of the bases of the trapezoid. Since the congruent trapezoids have equal areas, the area of the parallelogram is twice the area of the trapezoid.

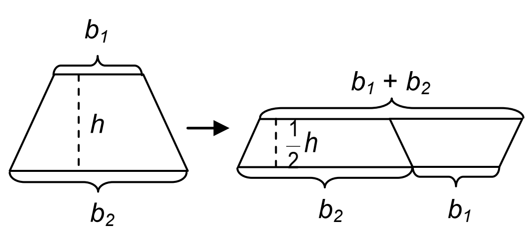
Method #2: Cut the trapezoid along a line parallel to a base and at half the height. Reassemble the two trapezoids as indicated to form a parallelogram. The height of the parallelogram will be half the height of the trapezoid, and the base of the parallelogram will be equal to the sum of the bases of the trapezoid. The new parallelogram will have the same area as the original trapezoid.

Method #3: Cut the trapezoid along one if its diagonals, forming two triangles. The height of the trapezoid will be equal to the height of each triangle, and each of the two bases of the trapezoid will be equal to a base of one of the two triangles.

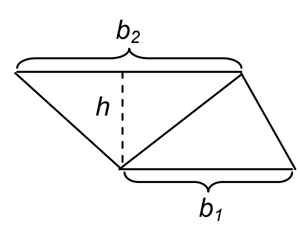
Method #1



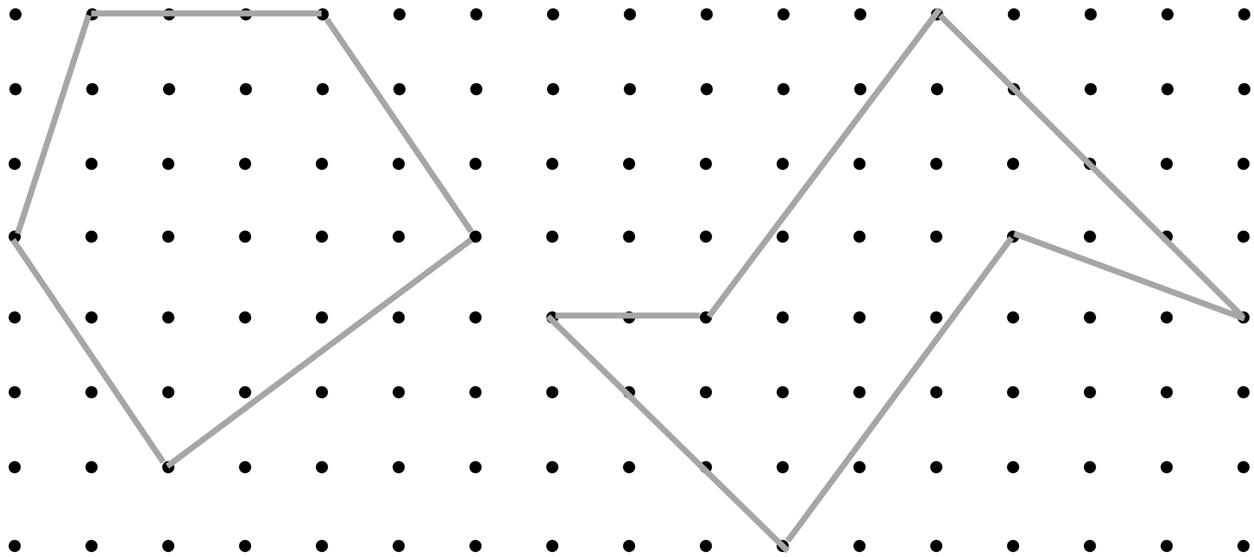

Method #2

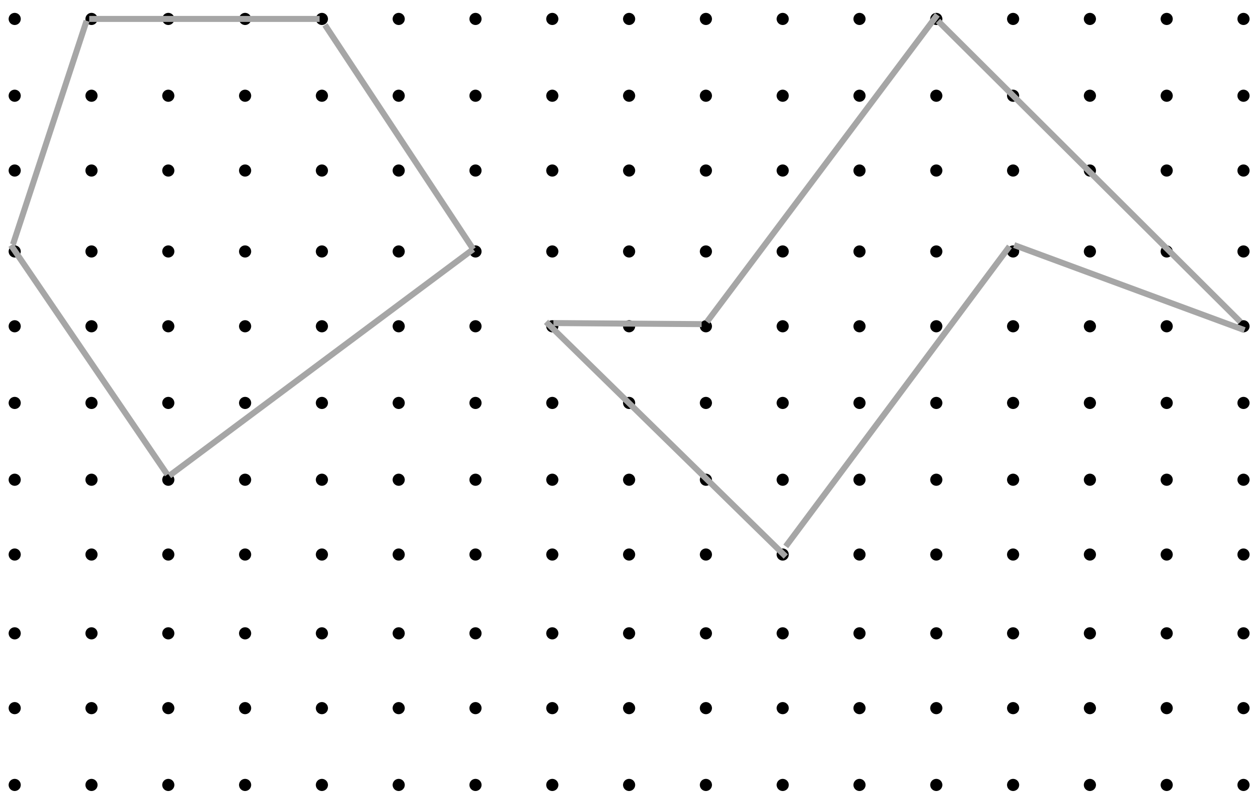



Method #3



# USING FORMULAS





# MEASURING TO FIND AREAS OF POLYGONS

For each problem:
- Identify the polygon and the corresponding area formula.
- Measure and label the relevant dimensions to the nearest tenth of a cm (mm).
- Write the appropriate area formula.
- Substitute values into the formula and evaluate to find the area.
- Use appropriate units in your answers.

1. Polygon name:

Area formula:

Substitute:

*A* = ____________________________

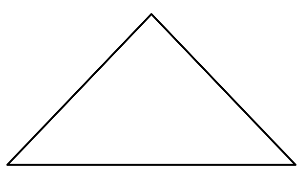
2. Polygon name:

Area formula:

Substitute:

*A* = ____________________________

3. Polygon name:

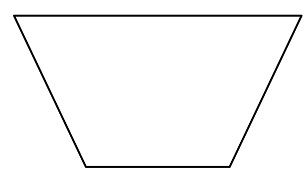


Area formula:

Substitute:

*A* = ____________________________

4. Polygon name:



Area formula:

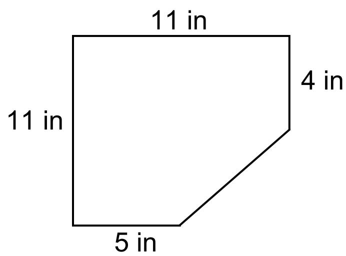
Substitute:

*A* = ____________________________

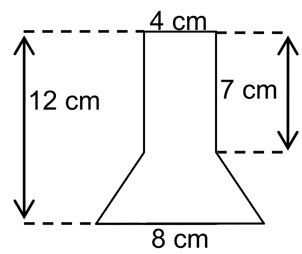
# AREAS OF COMPOSITE FIGURES

Below are several composite figures, formed from simple figures that do not overlap. Find the area of each composite figure below in square units. Assume any angle that appears to be a right angle is a right angle. Figures are not drawn to scale. Show your work and your answers clearly.

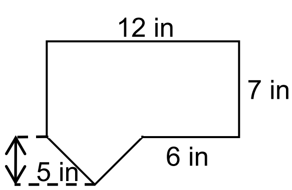
1.



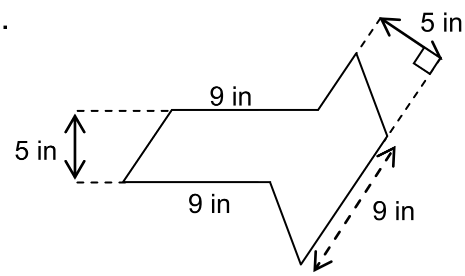

2.




3.




4.



# THE DISPLAY BOARD PROBLEM

This problem is about making a display board for a science fair project. You will use important geometry and measurement skills.

- The display board you create must be a 3 feet by 4 feet rectangle.
- The title sheet on the display board must be cut from an $8\frac{1}{2}$-inch by 11-inch piece of paper in the shape of an isosceles trapezoid. One base is $8\frac{1}{2}$ inches, and the other is 6 inches.
- You put a large picture in the center of the display board inside an isosceles triangle that is also cut from an $8\frac{1}{2}$-inch by 11-inch a piece of paper. The base of the triangle is 11 inches and the height is $8\frac{1}{2}$ inches.

1. Make a sample of your title sheet. Label all the measurements you will need to find its area.
2. Use graph paper to make a scale drawing of the display board with the title sheet. You may put the title sheet and picture wherever you like.
3. Find the area of the display board.
4. Find the area of the title sheet.
5. Find the area of the triangle with the picture on it.
6. Find the area of the remaining space you can use on the display board after the title sheet and triangle with the picture are attached.

# GEOMETRIC DESIGNS

This task requires you to use some rational number, measurement, and geometry concepts you've learned in the past. Choose the appropriate tools that you will need to effectively complete this problem.

1. Use a blank $8\frac{1}{2}$ inch $\times$ 11 inch piece of paper.

   a. Create four non-overlapping sections so that

      - A rectangle is 25% of the area of the paper,
      - A different rectangle is $\frac{3}{8}$ of the area of the paper,
      - A triangle is $\frac{1}{8}$ of the area of the paper, and
      - The last section includes the remainder of the area of the paper.

   b. Within each section, clearly label the relevant dimensions. Choose measurement tools appropriately.

   c. Compute each area to prove that your four sections were constructed correctly.

2. Use a different blank piece of paper (any size of your choice).

   a. Create another design that includes at least four different non-rectangular sections.

   b. Within each section, clearly label the relevant dimensions.

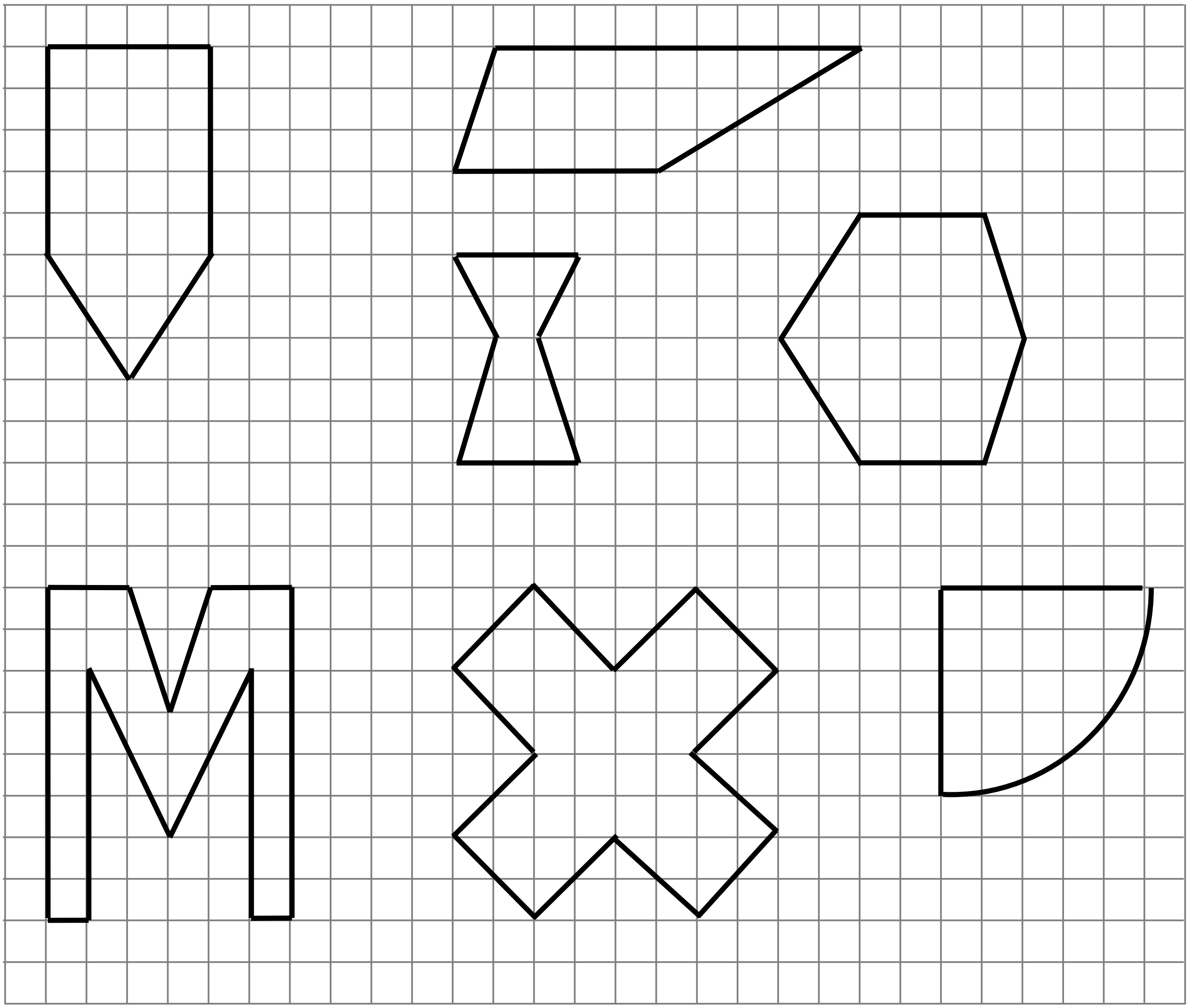
   c. Compute each area to prove that your sections total the area of the entire sheet of paper.

   d. Describe the sections using fractions, decimals, or percents.

Lightly color the sections on your papers to create mosaics if desired.

# AREA CHALLENGE

Find the area of each figure and justify your answer as directed by your teacher.



# AREA OF CIRCLES

| **Summary** | **Goals** |
|---|---|
| We will use our knowledge of the area of rectangles and circumference of circles to give an informal derivation of the area formula for a circle. We will use the formula to solve problems. | • Derive the area formula for circles.<br>• Solve problems that involve areas of circles. |

**Warmup**

Suppose the vertical and horizontal length between adjacent dots represents 4 feet.

| | |
|---|---|
| 1. Find the area of a 16 foot by 20 foot rectangle. Make a scale drawing of the rectangle. | 2. Find the circumference of a circle with a radius of 8 feet. (Your answer should be exact, with $\pi$ in it.) Make a scale drawing of the circle. |

# DERIVING THE AREA OF A CIRCLE

1. What is the formula for the circumference of a circle in terms of its radius *r*?

2. Your teacher will give you a paper circle and some directions for folding it, cutting it into wedges, and then arranging the wedges into a shape. Space is provided below to tape, glue stick, or sketch the shape.

Use your knowledge of the area of a parallelogram / rectangle, and the circumference of a circle to find the area of this shape.

3. This figure, before folding and cutting, was a ______________________________,

   now looks much like a ______________________________, and the more wedges

   you make, the closer it gets to becoming a ______________________________.

4. The "base" of the figure is approximately _________________________________ of the circle.

5. The height of the figure is approximately _________________________________ of the circle.

6. Write an equation for the approximate area of the figure.

7. Substitute the expression for the circumference of the circle from problem 1 above into your formula and simplify.

8. What is the formula for the area of a circle, in terms of the radius?

   $A_{\bigcirc}$ = ______________

# AREA REPRESENTATIONS

1. Use the formula for the area of a circle to complete the table to the right.

| **Radius** (***x***) | **Area** (***y***) |
|---|---|
| 0 units | |
| 1 units | |
| 2 units | |
| 3 units | |
| 4 units | |

2. Draw horizontal and vertical axes on the graph below. Label and scale the axes. Title the graph.

3. Graph the data points. How can you tell that a line does not fit the data very well?

4. What does the point (0, 0) represent on this graph?

5. What does the point (1, *y*) represent on this graph?

6. Write an equation that best represents this graph.

7. Explain in words what this equation means.

8. Explain why each of the following does NOT represent a proportional relationship.

   a. The values in your data table.

   b. The graph.

   c. The equation.

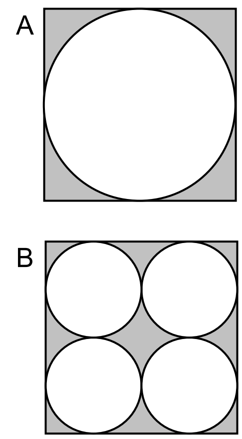
# CIRCLE PROBLEMS

| | |
|---|---|
| 1. The exact circumference of a circle is $8\pi$ cm. Find the exact area. | 2. The exact area of a circle is $25\pi$ cm$^2$. Find the exact circumference. |

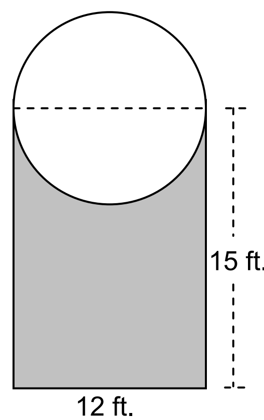
Squares A and B to the right both have side length equal 4 cm. All four circles in square B have the same diameter length.

3. Predict which square has the greater amount of white space. ______

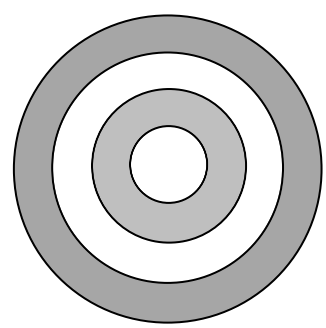
4. Test your prediction by calculating the area of the white space in both squares.




5. Find the shaded area of the basketball key to the right.




6. A dartboard is a representation of <u>concentric</u> circles, which are circles that have the same center. The smallest circle (the bullseye) has diameter $d$ = 4 in. Each successive circle has a radius 2 inches greater than the previous one. What fraction of the whole dartboard is the smaller gray ring?



## MATH NOTES (Continued)

### The Number Pi

Pi (usually written as the Greek letter $\pi$) is the ratio of the circumference of a circle to its diameter. The constant $\pi$ is slightly greater than 3, so that the circumference of a circle is a little more than 3 times its diameter.

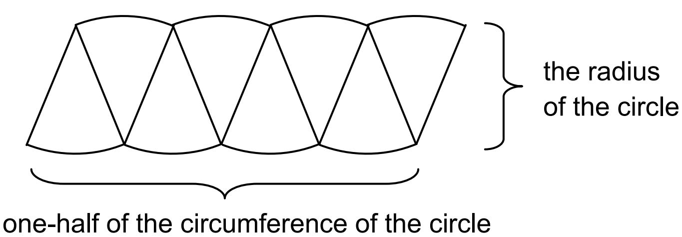
Various approximations to the ratio of the circumference to the diameter of a circle have been used by many civilizations over the centuries. The oldest written approximations to $\pi$, dating from 1600-1900 BCE and accurate to within 1%, come from Babylonic cuneiform tablets and an Egyptian papyrus (the Rhind papyrus).

The Greek mathematician Archimedes (third century BCE) found some very accurate approximations to $\pi$. In his manuscript *Measurement of a Circle*, Archimedes showed that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. We often use the approximation $3\frac{1}{7} = \frac{22}{7}$ for the value of $\pi$ today. Other common approximations to $\pi$ are 3.14 and 3.1416.

The notation $\pi$ for pi was first used around 1700. It probably was meant to stand for "periphery." The notation was adopted by Euler in the mid-1700s, and subsequently came into common usage. That same century, the Swiss mathematician J.H. Lambert established that $\pi$ is not a rational number, that is, $\pi$ is not a quotient of integers. Finally, in 1882, a German mathematician Ferdinand Lindemann succeeded in proving that $\pi$ is a transcendental number, that is, $\pi$ is not the solution of any algebraic equation with integer coefficients. By doing this, he settled the last of the three classical Greek problems, showing that it is impossible to "square a circle" (construct a square of the same area as a given circle) using a ruler and compass.

### Area of a Circle

There is a simple informal derivation of the formula for the area of a circle that students can perform with paper and scissors. Start with any circle and cut it into equal sized wedges. Then alternate the tips to form a shape that resembles a parallelogram.




If desired, cut one wedge again so that the shape more closely resembles a rectangle. Or cut all the wedges in half again. The more times the wedges are halved, the closer the shape comes to be a rectangle.

The base of this "rectangle" is approximately half the circumference $C$ of the circle, and the height of this "rectangle" is approximately the radius $r$ of the circle. Since the area of a rectangle is (base) $\bullet$ (height), the area $A$ of the circle is given approximately by

$$A \approx \text{(base)} \bullet \text{(height)} \approx \left(\frac{1}{2}C\right) \bullet r .$$

In the limit, as the approximations become more exact, we obtain $A = \frac{1}{2}C \bullet r$. Since $C = 2\pi r$, we obtain

$A = \frac{1}{2} \bullet 2\pi r \bullet r$ or $A = \pi r^2$.

# CIRCLES