MAKING SENSE OF AREA AND VOLUME IN MIDDLE SCHOOL MATHEMATICS

Presented by
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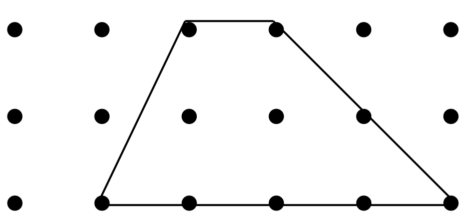
(Session 451, Poll Code 12028)

The Center for Mathematics and Teaching

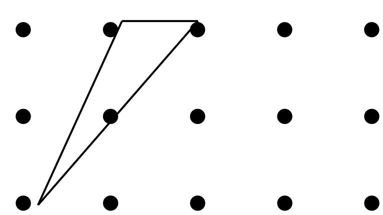
www.mathandteaching.org

While you are waiting:



A = ______

Why?

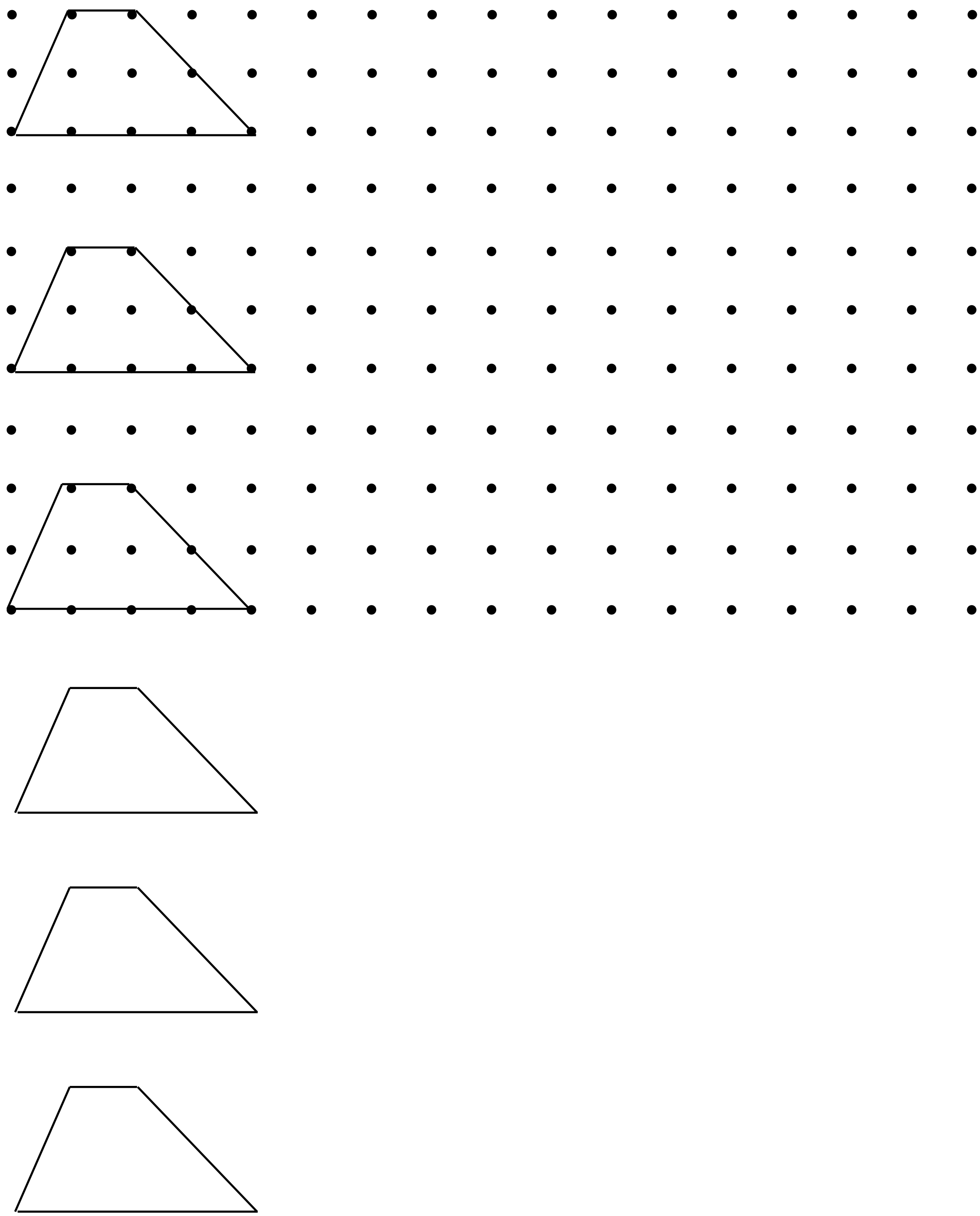


A = ______

Why?

AREA OF A TRAPEZOID

(Grade 6)



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AREA OF A CIRCLE

(Grade 7)

1. What is the formula for the circumference of a circle in terms of its radius *r*?

2. Your teacher will give you a paper circle and some directions for folding it, cutting it into wedges, and then arranging the wedges into a shape. Space is provided below to tape, glue stick, or sketch the shape.

Use your knowledge of the area of a parallelogram / rectangle, and the circumference of a circle to find the area of this shape.

3. This figure, before folding and cutting, was a ______________________________,

   now looks much like a ______________________________, and the more wedges

   you make, the closer it gets to becoming a ______________________________.

4. The “base” of the figure is approximately _________________________________ of the circle.

5. The height of the figure is approximately _________________________________ of the circle.

6. Write an equation for the approximate area of the figure.

7. Substitute the expression for the circumference of the circle from problem 1 above into your formula and simplify.

8. What is the formula for the area of a circle, in terms of the radius?

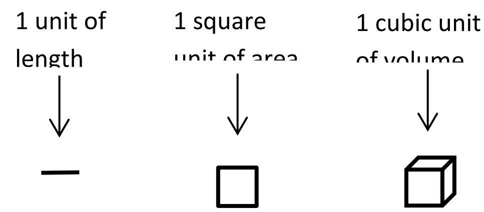
Area of a Circle
A = ______________

VOLUME OF A CYLINDER

(Grade 8)

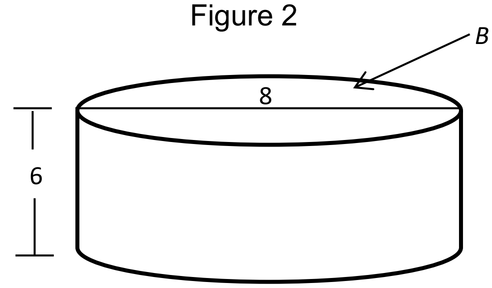
1. Figure 1 illustrates some common units of measurement. The line segment represents one unit of _______________, the square represents one square unit of _______________, and the cube represents one cubic unit of _____________.

Figure 1



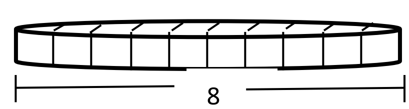
2. Figure 2 illustrates a shape that is called a _______________. It has a base area (B) in the shape of a _______________. The _________________ of the base is 8 units.
Find the area of the base. (Use $\pi = 3.14$.)

Figure 2



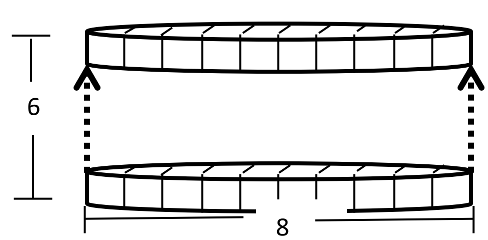
3. Figure 3 illustrates the bottom "layer" of the cylinder, which has height 1 unit. How many cubic units are in this layer? In other words, what is the volume of the bottom layer?

Figure 3



4. Figure 4 suggests that the cylinder has ______ layers. How many cubic units are in this cylinder? In other words, what is the volume of the cylinder?

Figure 4



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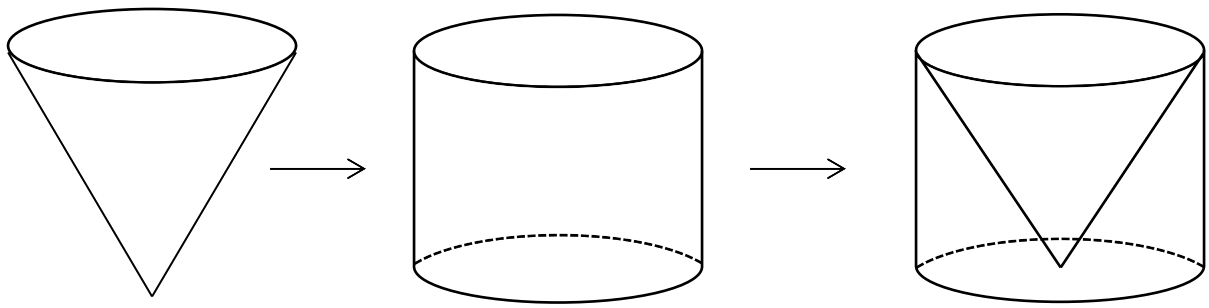
VOLUME OF A CONE

(Grade 8)



1. Compare the height *h* of the cone and the cylinder.

2. Compare the areas of the circular bases *B* of the cone and the cylinder.

3. Predict the number of pours it will take from the cone to fill the cylinder. ________

4. I think the volume of the cone is ________ of the volume of the cylinder.
   fraction

(After the experiment)

5. Derive the formula for volume of a cone.

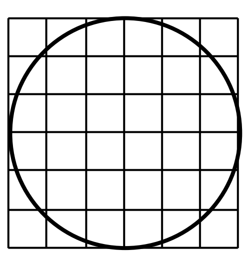
   a. Begin with volume of a cylinder ⟶ $V_{cylinder}$ = ______________

   b. The cone is ________ of the cylinder ⟶ V_{cone} = (_____) • $V_{cylinder}$
   fraction

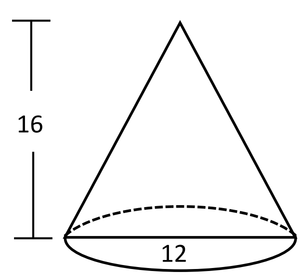
   c. Substitute ⟶ V_{cone} = ________________

Find the volume of each cone described below. (Use $\pi = 3.14$.)

6. The base area picture below.
   Each small square is 1 square unit.
   Each height is 10 units



7. units given in cm



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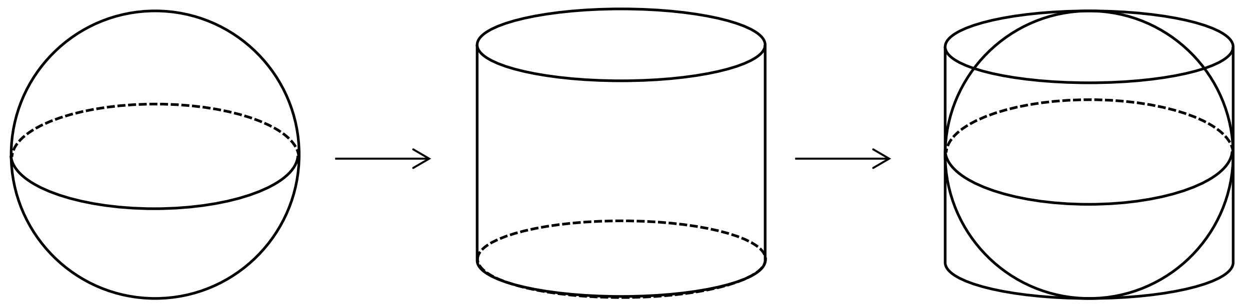
VOLUME OF A SPHERE

(Grade 8)



1. Compare the height h of the cylinder to the diameter of the sphere.

2. Compare the diameter d of the cylinder to the diameter of the sphere.

3. Predict the number of pours it will take from the sphere to fill the cylinder. ________

4. I think the volume of the sphere is ________ of the cylinder.
   fraction

(After the experiment)

5. Derive the formula for volume of a sphere.

   a. Begin with volume of a cylinder → $V_{cylinder}$ = ________________

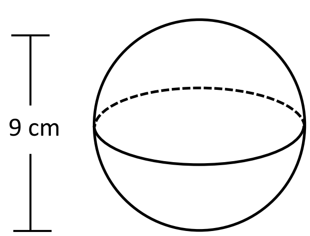
   b. The sphere is ________ of the cylinder → V_{sphere} = (______) • $V_{cylinder}$
      fraction

   c. Substitute → V_{sphere} = ________________
      (recall the relationship between the height of the cylinder and diameter of the sphere)

      V_{sphere} = ________________

      V_{sphere} = ________________

6. Find the volume of the sphere. (Use $\pi =$ 3.14.)



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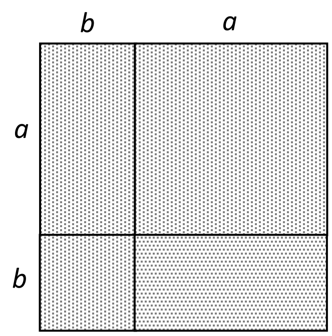
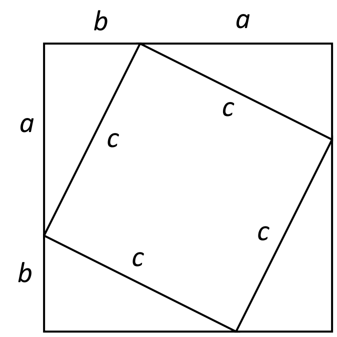
ONE MORE PROOF

(Grade 8)



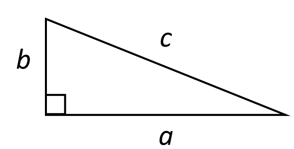


1. Write the areas inside the polygonal pieces in the two square figures above. Use the information from the pieces previously cut.

2. Write an equation that equates the sum of the areas of the shaded polygons with the sum of the areas of the unshaded polygons.

3. Simplify the equation.

4. Use words to state the meaning of this equation as it refers to the legs and the hypotenuse of the original triangle.



5. This relationship is called the ____ .

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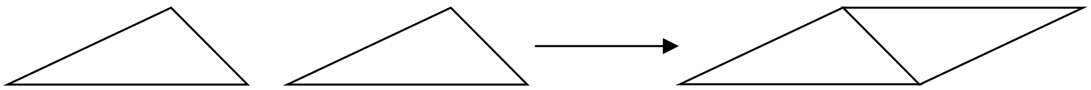
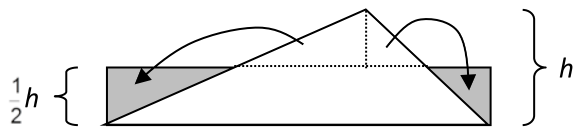
The Area of a Triangle by Duplication and Dissection

Method #1: Duplicate the triangle (create a congruent triangle), and reassemble the two congruent triangles as indicated to form a parallelogram. Since congruent triangles have equal areas, the area of the triangle is half the area of the parallelogram.



Method #2: Orient triangle with the longest side as the base. Cut the triangle along a line parallel to the base and at half the height. Cut the top triangular piece along the perpendicular to the base from the top vertex. The two resulting pieces from the top can be positioned over the shaded pieces of the rectangle with the same base as the triangle and half the height, so that the three pieces from the triangle cover the rectangle exactly once.



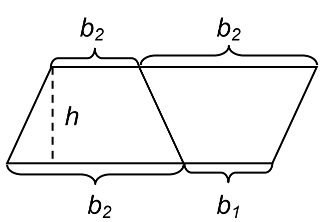
The Area of a Trapezoid by Duplication and Dissection

Method #1: Duplicate the trapezoid (create a congruent trapezoid), and reassemble the two congruent trapezoids as indicated to form a parallelogram. The parallelogram will have the same height as the trapezoid, and the base of the parallelogram will be equal to the sum of the bases of the trapezoid. Since the congruent trapezoids have equal areas, the area of the parallelogram is twice the area of the trapezoid.

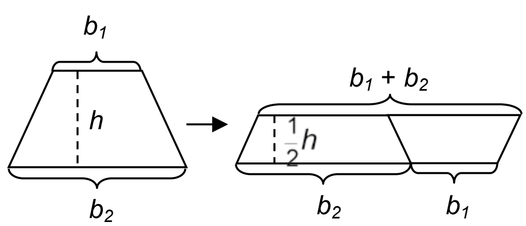
Method #2: Cut the trapezoid along a line parallel to a base and at half the height. Reassemble the two trapezoids as indicated to form a parallelogram. The height of the parallelogram will be half the height of the trapezoid, and the base of the parallelogram will be equal to the sum of the bases of the trapezoid. The new parallelogram will have the same area as the original trapezoid.

Method #3: Cut the trapezoid along one if its diagonals, forming two triangles. The height of the trapezoid will be equal to the height of each triangle, and each of the two bases of the trapezoid will be equal to a base of one of the two triangles.

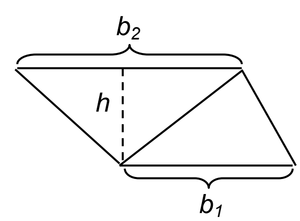
Method #1



Method #2

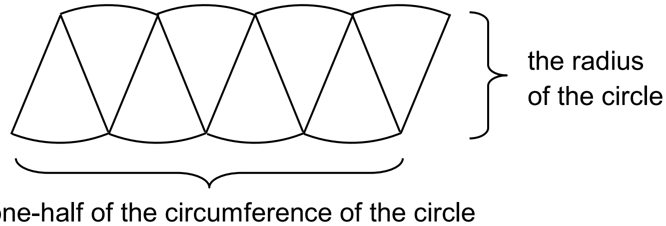


Method #3

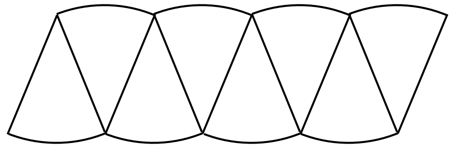


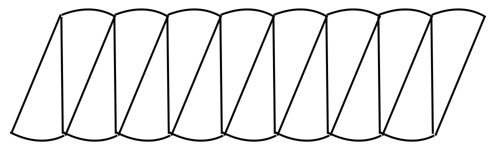
Area of a Circle

There is a simple informal derivation of the formula for the area of a circle that students can perform with paper and scissors. Start with any circle and cut it into equal sized wedges. Then alternate the tips to form a shape that resembles a parallelogram.



If desired, cut one wedge again so that the shape more closely resembles a rectangle. Or cut all the wedges in half again. The more times the wedges are halved, the closer the shape comes to be a rectangle.





The base of this "rectangle" is approximately half the circumference C of the circle, and the height of this "rectangle" is approximately the radius r of the circle. Since the area of a rectangle is (base) $\bullet$ (height), the area A of the circle is given approximately by

$$A \approx \text{(base)} \bullet \text{(height)} \approx \left(\frac{1}{2}C\right) \bullet r .$$

In the limit, as the approximations become more exact, we obtain $A = \frac{1}{2}C \bullet r$.

Since $C = 2\pi r$, we obtain $A = \frac{1}{2} \bullet 2\pi r \bullet r$ or $A = \pi r^2$.

When Is a Proof a Proof?

The argument in the note above for the formula for the area of a circle is certainly convincing. Is it a proof? Yes and no. It does not constitute a formal proof, yet it contains the essential line of reasoning of a correct proof. A person with a sophisticated background in mathematical analysis (epsilons and deltas) may look at the argument and say that yes, it is a proof, in the sense that he or she can translate to the language of the analyst to form a correct formal proof.

Even mathematicians rarely produce complete formal proofs of their theorems. Hyman Bass ("Mathematics and Teaching," in *I, Mathematician*, edited by P. Cassazza, S,G.Krantz, and R.D. Ruden, MAA, 2015) suggests:

> "Proving a claim is, for a mathematician, an act of producing, for an audience of peer experts, an argument to convince them that a proof of the claim exists."

This statement sheds light on how we might treat mathematical reasoning and proof in the early grades—what we might expect, and what we should not expect.

In any event, the idea of proof is a flexible notion. The conventions and standards of the mathematical community, whether that community is the group of students in a classroom or the readers of a mathematics research journal, play a role in deciding whether an argument is a valid proof.

Volume Formulas

Recognizing the general structure of formulas for prisms, pyramids, cylinders, cones, and spheres makes it easier to understand and remember them. Here is a grouping of the formulas that may prove useful.

Suppose that all horizontal planes cut a solid of height h in a figure that has the same area B as the base of the solid. Then the volume of the solid is simply the product of its height and the area of the base, $V = Bh$.

Volume of a Rectangular Prism	**Volume of a Cylinder**
Let ℓ = length and w = width of rectangular base.	Let r = radius of the circular base.
$V = B h$	$V = B h$
Area of base (B) $= \ell w$	Area of base (B) $= \pi r^2$
Therefore, $V = \ell w h$	Therefore, $V = \pi r^2 h$

Through experimentation, observe that the volume of a cone is $\frac{1}{3}$ of the volume of a cylinder with the same altitude and base. This relationship also holds for pyramids. More generally, to find the volume of figures with one base and an apex, multiply the area of the base (B) by $\frac{1}{3}$ of the height (h) of the figure.

Volume of a Rectangular Pyramid	**Volume of a Cone**
Let ℓ = length and w = width of rectangular base.	Let r = radius of the circular base
$V = \frac{1}{3} B h$	$V = \frac{1}{3} B h$
Area of base (B) $= \ell w$	Area of base (B) $= \pi r^2$
Therefore, $V = \frac{1}{3} \ell w h$	Therefore, $V = \frac{1}{3} \pi r^2 h$

Through experimentation, observe that the volume of a sphere is $\frac{2}{3}$ of the volume of a cylinder whose diameter and height are the same as the diameter of the sphere. Use substitution to derive the formula of a sphere.

Volume of a Sphere

Let r = radius of the sphere and cylinder
Then height (h) of cylinder = $2r$
Volume of cylinder = $\pi r^2 (2r) = 2\pi r^3$

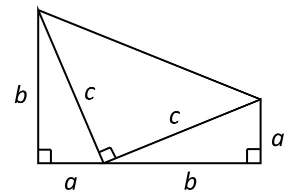
Observe that volume of sphere is $\frac{2}{3}$ of the volume of a cylinder.

Therefore, $V_{sphere} = \frac{2}{3} \cdot 2\pi r^3 = \frac{4}{3}\pi r^3$

President Garfield's Proof of the Pythagorean Theorem

The proof in this lesson is only one of more than 400 proofs of the Pythagorean Theorem that have been recorded. Many of the proofs are simple variations of another proof.

Here is another proof, discovered in 1876 by President James A. Garfield while a member of the House of Representatives. Garfield was also a mathematics teacher.



Let A be the area of the entire figure, which is a trapezoid. The area of the trapezoid is

$$A = \frac{(a+b)(a+b)}{2} = \frac{a^2+2ab+b^2}{2}.$$

Since the area of the trapezoid is the sum of the areas of the three triangles, we have

$$A = \frac{1}{2}(ab) + \frac{1}{2}(c^2) + \frac{1}{2}(ab) = \frac{c^2+2ab}{2}.$$

Equating the two expressions for A, we obtain

$$\frac{a^2+2ab+b^2}{2} = \frac{c^2+2ab}{2}.$$

Hence, $a^2+b^2=c^2$.

Some Online Resources

Volume of a cone

youtube.com/watch?v=QnVr_x7c79w

Volume of a sphere

youtube.com/watch?v=aLyQddyY8ik

Area of a circle (standard proof – includes Spanish option)

https://www.youtube.com/watch?v=YokKp3pwVFc

Area of a circle (pretty clever)

https://www.youtube.com/watch?v=whYqhpc6S6g

STANDARDS FOR MATHEMATICAL CONTENT	
6.EE.A	**Apply and extend previous understandings of arithmetic to algebraic expressions.**
6.EE.3	Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*
6.G.A	**Solve real-world and mathematical problems involving area, surface area, and volume.**
6.G.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
7.EE.A	**Use properties of operations to generate equivalent expressions.**
7.EE.2	Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, a + 0.05a = 1.05a means that "increase by 5%" is the same as "multiply by 1.05."*
7.G.B	**Solve real-life and mathematical problems involving angle measure, area, surface area, and volume**.
7.G.4	Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
8.G.B	**Understand and apply the Pythagorean Theorem.**
8.G.6	Explain a proof of the Pythagorean Theorem and its converse.
8.G.C	**Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.**
8.G.9	Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.

STANDARDS FOR MATHEMATICAL PRACTICE

Here are a few examples of how the Standards for Mathematical Practice are applied in this presentation

MP2 Reason abstractly and quantitatively. Students establish formulas for polygons by generalizing from examples. Students use algebraic reasoning to establish the formula for the area of a circle and the Pythagorean theorem.

MP5 Use appropriate tools strategically. [Students may cut out and manipulate dot paper drawings of polygons to reason about their areas. Experiments pouring water helpstudents understand volume formulas for the pyramid, cone, and sphere.

MP7 Look for and make use of structure. Students use their knowledge of the area of a rectangle to derive the parallelogram area formula and the area of a parallelogram to derive the triangle and trapezoid area formulas. Students observe the relationship between a prism and cylinder to establish the formula for the volume of a cylinder.

MP8 Look for and express regularity in repeated reasoning. As students develop the formula for the volume of a right rectangular prism, call attention to the pattern emphasized through questioning, namely that the volume is the product of the volume of the top layer and the number of layers. Students use similar reasoning to establish volume formulas for pyramids and cones based on formulas for prisms and cylinders.

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