WHAT'S NEW IN CCSS MIDDLE SCHOOL PROPORTIONAL REASONING?

Presented by

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EXAMPLE FOR SESSION EVALUATIONS

38102 is the sample. DO NOT use it for this session

Please use poll code 45026

GOALS FOR THIS PRESENTATION

1. Introduce and/or clarify some of the new CCSS content standards in the domain of ratios and proportional relationships.

2. Discuss interpretations of these standards, and the vocabulary contained within, based upon "the progressions." *

3. Provide examples of some of the representations named in the standards.

4. Discuss coherence and connections through the middle grades in this topic area, especially in grades 6-7.

5. Connect to the Standards for Mathematical Practice

* Progressions for the Common Core State Standards in Mathematics (referred to by many as "*the progressions*") (Draft, 6-7 Progression on Ratios and Proportional Relationships, by the Common Core Standards Writing Team) http://ime.math.arizona.edu/progressions/

WHERE WE WERE
From the CMC Communicator, 2008 (Goldstein)

"Proportional reasoning is both the capstone of elementary arithmetic and the cornerstone of all that is to follow. It therefore occupies a pivotal position in school mathematics (and science) programs." - Lesh, R., Post, T., & Behr, M. (1988).

A typical textbook treatment of proportionality tends to progress like this:

- Define ratio, rate, and proportion.
- Express that if $\frac{a}{b} = \frac{c}{d}$, then ad = bc. Typically there is no proof or explanation of why this is true, however, it is usually followed by empirical evidence. (i.e. $\frac{2}{5} = \frac{4}{10} \Rightarrow 2 \bullet 10 = 4 \bullet 5$)
- Do problems using a proportion (percent , maps, scale drawings, similar figures, mixtures).

However, proportional reasoning has a far greater scope and purpose than to know only the above. Proportional reasoning includes:

- multiplication, division, and the inverse relation between the two;
- the need for non-integer values (fraction, decimal, and percent representations);
- ratio and rate;
- proportion as a tool for solving problems;
- linear functions in the form $y = kx$, where k is the constant of proportionality (the graph of which is a line through the origin).

It is suggested that these concepts should be taught in an integrated fashion over a long period of time through experiences with a variety of situations. Proportional reasoning has many facets and must involve more than using procedures to find missing values or to compare quantities.

Where are we headed?

MATHEMATICAL CONTENT STANDARDS
A Selection from Grades 6-7

6.RP.1	Understand the concept of a ratio and use **ratio language to describe a ratio relationship between two quantities**. *For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."*
6.RP.2	Understand the concept of a **unit rate** a/b associated with a ratio $a:b$ with $b \neq 0$, and **use rate language in the context of a ratio relationship**. *For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."*
6.RP.3a	Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about **tables of equivalent ratios, tape diagrams, double number line diagrams, or equations**: Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
6.RP.3b	Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about **tables of equivalent ratios, tape diagrams, double number line diagrams, or equations**: Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?*
6.RP.3c	Find a **percent of a quantity as a rate per 100** (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving **finding the whole, given a part and the percent**.
7.RP.A.3	Use proportional relationships to **solve multistep ratio and percent problems**. Examples: simple interest, tax, **markups and markdowns**, gratuities and commissions, fees, **percent increase and decrease**, percent error.

Adapted from *MathLinks*

MATHEMATICAL CONTENT STANDARDS

(Continued)

The Standards for Mathematical Content outline the skills and understandings students will learn by grade level or course, and are organized around domains and coherent clusters.

<table>
<tr><td>K</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td colspan="9">Geometry</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="3">Statistics and Probability</td></tr>
<tr><td colspan="6">Numbers and Operations in Base Ten</td><td colspan="3">The Number System</td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td></tr>
<tr><td>Counting and Cardinality</td><td colspan="2"></td><td colspan="3">Number and Operations: Fractions</td><td colspan="2">Ratios and Proportional Relationships</td><td>Functions</td></tr>
</table>

THE CCSS SHIFTS

Seeking increased FOCUS, COHERENCE, and RIGOR

Focus	**Coherence**	**Rigor**
• Strongly emphasized at appropriate grade levels for deep understanding. • Fewer topics and standards per grade level allow for greater depth and focus; avoids the rush to "cover" content, and is counter to the "mile-wide-inch-deep" approach.	• Thinking <u>across</u> grade levels, and linking to major topics <u>within</u> grade levels. • Important ideas can be linked across any one grade level (horizontally), and also from one grade level to the next (vertically).	• Pursue, with equal intensity in major topics: (1) conceptual understanding; (2) procedural skill and fluency; (3) problem solving and application. • The recent culture in mathematics education was to concentrate mainly on procedural skill and fluency (think standardized testing).

Adapted from *MathLinks*

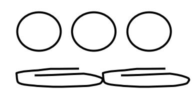
SOME VOCABULARY

AS INTERPRETED FROM "THE PROGRESSIONS"

A ratio associates two or more quantities.

The ratio of a to b can be denoted by $a : b$ (read " a to b "). Also stress the language *for each*, *for every*, *per*, etc.

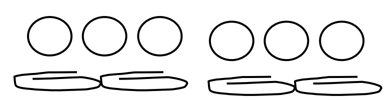
The ratio of coins to paperclips in the picture is 3 to 2.



The value of this ratio can be written as the quotient $\frac{3}{2}$.

This value is connected to the concept of rate. Fractions are downplayed when discussing ratio.

The original ratio of coins to paperclips (3 to 2) and the new ratio of coins to paperclips are equivalent ratios because each measurement in the original ratio pair can be multiplied by the same number to arrive at the new ratio pair (6 to 4). We say that equivalent ratios are in a proportional relationship.



A ratio has an associated rate, which includes a numerical value with the units.

Example: If you drive 40 miles in 2 hours, then the ratio 40 miles for every 2 hours can be written $\frac{40}{2}$ miles for every 1 hour, or 20 miles for every 1 hour. This is the rate.

A unit rate is the numerical part of the rate (to highlight the "for every 1.")

Example: For the rate above, $\frac{40}{2}$ miles for every hours is equivalent to 20 miles per 1 hour, so the unit rate is $\frac{40}{2}$, or more simply 20.

Let's see how some of this might play out…

Adapted from *MathLinks*

RATIO TABLES AND TAPE DIAGRAMS

(Grade 6)

A <u>ratio table</u> is a table whose entries are equivalent ratios. A ratio table illustrates a proportional relationship.

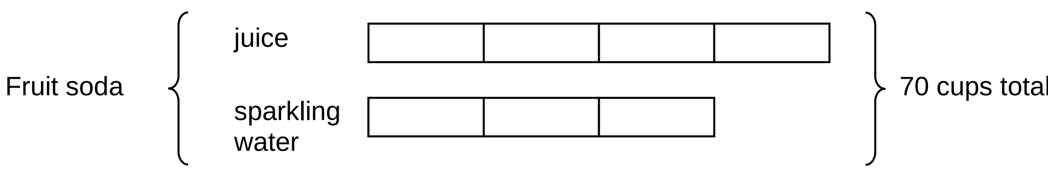
Milo likes to make fruit soda when he has people over to his house. He has determined that the juice to sparkling water ratio should be 4 : 3.

1. Fill in the ratio table for this information

Cups juice	4			200	2
Cups sparkling water		12			
Total cups			14		

A <u>tape diagram</u> for two quantities is two strips that visually depict the number of parts in each quantity. Tape diagrams are typically used when the quantities have the same units.

2. Milo estimates that he will want 70 cups in all. How much juice and how much sparkling water will he need?



3. Kendra makes tie-dyed shirts. For the orange dye she uses red and yellow in a ratio of 3 : 2. How many quarts of red and yellow dye will she need if she wants to make 80 quarts of dye? Use a tape diagram to solve.

Adapted from *MathLinks*

RATIO TABLES AND DOUBLE NUMBER LINES

(Grade 6)

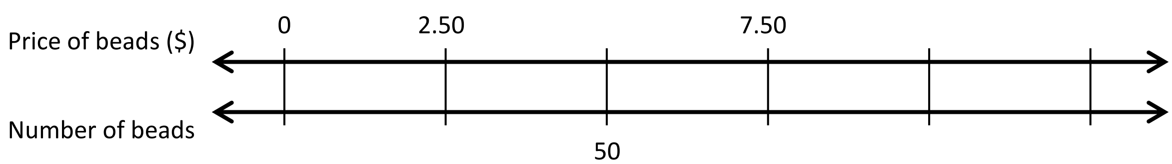
1. Why is this a ratio table?

   What is the unit rate in price per bead?

	Store A	Store B	Store C	Store D	Store E
Price of beads	$5.00	$2.50	$10.00	$7.50	$12.50
Number of beads	50	25	100	75	125

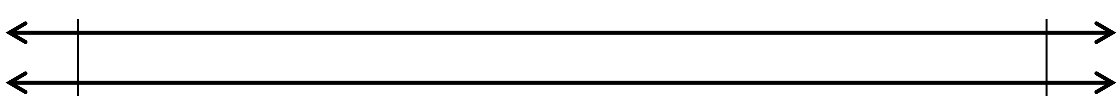
> A <u>double number line</u> is a diagram made up of two parallel number lines that visually depict the relative sizes of two quantities. Double number lines are often used when the two quantities have different units.

2. Complete the double number line for the table above.



3. Alberto jogs 5 yards every 2 seconds. Complete the ratio table and make a double number line diagram. Hint: a few entries may not fit easily, so make a case for which you may want to leave out.

yards	5			20			1	2
seconds		4	6		0	1		



Where does Alberto's unit rate in yards per second appear in the table?

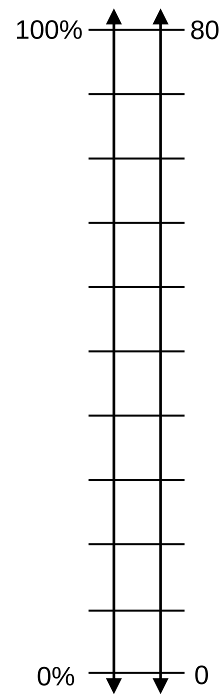
Use the unit rate to find the number of yards he can jog in one minute.

Adapted from *MathLinks*

USING DOUBLE NUMBER LINE DIAGRAMS TO UNDERSTAND PERCENT PROBLEMS

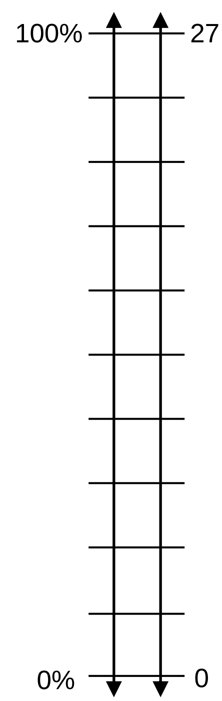
(Grade 6)

Number the double number line diagram. Then complete the chart.



	Words	**Equivalent fractions**
Ex.	50% of 80 is 40.	$\frac{50}{100} = \frac{40}{80}$
1.	40% of 80 is _____.	
2.		$\frac{60}{100} = \frac{\square}{80}$
3.	75% of 80 is _______.	
4.		$\frac{25}{100} = \frac{\square}{80}$

5. Explain why 27% of 80 might be difficult to find on the double number line diagram above.



6. Use the double number line diagram to the right to find 80% of 27. Hint: first think about what each dash on the right must be numbered by.

Use multiplication to find the following.

7. 27% of 80	8. 80% of 27

Adapted from *MathLinks*

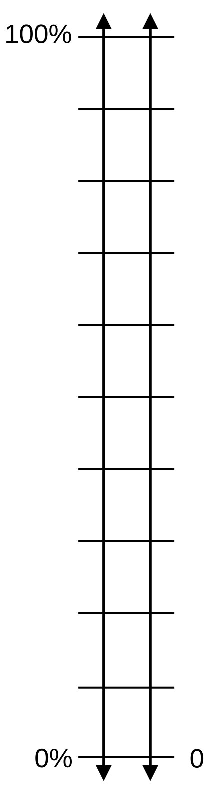
FIND THE MISSING VALUES

(Grade 6)

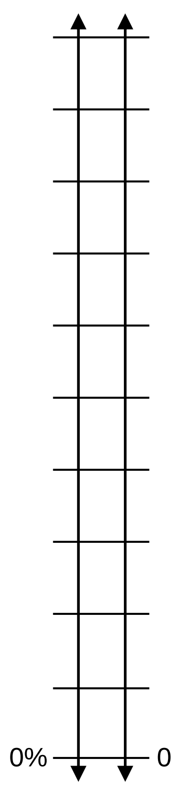
Use the double number lines to help you write equivalent fractions to represent each percent problem. Then answer the questions. Remember : $\frac{part_1}{100} = \frac{part_2}{whole}$

Be careful, it's not only $part_2$ that is missing in each problem.

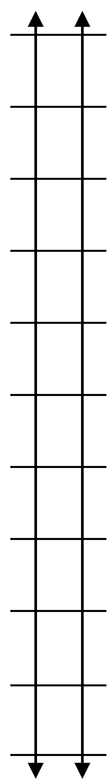
6. What is 40% of 50?



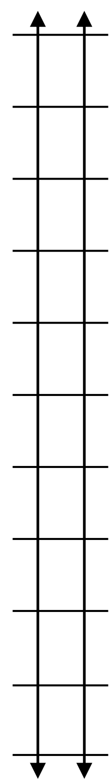
7. 50 is 25% of what?



8. What is 15% of 400?



9. 100 is 20% of what?



10. Use any method: 80% of what number is 60?

PROPORTIONS: GIVE THE REASONS

(Grade 7)

1. Write what was done for each step a-f.

- Fact: if $\frac{m}{p} = \frac{n}{p}$, then $m = n$ (referred to as the numerator equality property).

Equation/Steps	State what was done
(a) $\frac{x}{-3} = \frac{3}{8}$	(a) given equation (proportion)
(b) $\frac{x}{-3} \bullet \frac{8}{8} = \frac{3}{8} \bullet \frac{-3}{-3}$	(b)
(c) $\frac{8 \bullet x}{-24} = \frac{3 \bullet (-3)}{-24}$	(c)
(d) $8x = -9$ (e) $\frac{8x}{8} = \frac{-9}{8}$	(d) (e)
(f) $x = \frac{-9}{8}$	(f)

2. Look carefully at (a) $\frac{x}{-3} = \frac{3}{8}$ and (d) $8x = -9$. Going from (a) to (d) in one step is a process called cross multiplication.

   Rewrite the equation $-\frac{7}{3} = \frac{5}{x}$ using cross multiplication: ________ = ________

Use this process to solve each equation (proportion).

3. $-\frac{4}{9} = \frac{5}{x}$	4. $\frac{x}{6} = \frac{-8}{5}$	5. $\frac{3}{7} = -\frac{x}{5}$

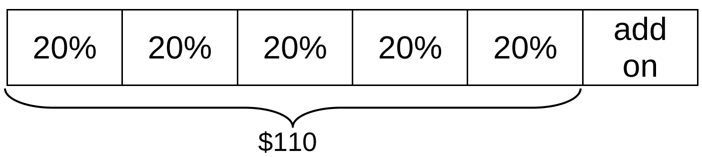
Adapted from *MathLinks*

MORE PERCENT PROBLEMS

(Grade 7)

1. Hans and Franz are pricing new skateboards and they each have $100 to spend. They find one they both like for $110, so they know they each need to save more. The store owner says that the price of this popular style may go up as much as 20% in the future. They want to figure out how expensive this skateboard might possibly become.

   Hans started to draw a tape diagram to figure out the potential price increase, and Franz set up a proportion. Study their work and then finish what they started to find the potential new price.



$$\frac{\text{price now}}{\text{after increase}} \rightarrow \frac{\$110}{x} = \frac{100\%}{120\%}$$

   a. Finish Hans' work.

   b. Finish Franz's work

2. Dieter is skateboard shopping at the same store and is faced with the same potential 20% price increase. If he waits to buy a $140 skateboard, how expensive could it get? Use both methods above and show all work.

3. Hans and Franz went to a different store and got a pleasant surprise. The skateboard they wanted was on sale for $90, which was a 20% discount off the original price. What was the price before the discount? Use both methods above and show all work.

   c. Tape diagram (Hans' method)

   d. Proportion equation (Franz's method)

Adapted from *MathLinks*

STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe a variety of processes and proficiencies that educators seek to develop in mathematically proficient and fluent students across all grade levels.

Many processes and proficiencies in these practice standards overlap, several may be used together on any given problem or task, and rarely would we expect students to use them all at once. We do expect that over time students will use them frequently. In addition, some will be used naturally within the context of solving particular problems, and others will only occur in an environment in which students are provided ample opportunities to collaborate and discuss.

One way to think about the practices is in groupings (graphic from CCSS-M author, Bill McCallum).

HABITS OF MIND

MP1 Make sense of problems and persevere in solving them
MP6 Attend to precision

REASONING AND EXPLAINING

MP2 Reason abstractly and quantitatively
MP3 Construct viable arguments and critique the reasoning of others

MODELING AND USING TOOLS

MP4 Model with mathematics
MP5 Use appropriate tools strategically

SEEING STRUCTURE AND GENERALIZING

MP7 Look for and make use of structure
MP8 Look for and make use of repeated reasoning

From CCSS-M:

“Students who lack understanding of a topic may rely on procedures too heavily...In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices. In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. Without understanding, a student may rely on procedures and may not represent problems coherently, justify conclusions, apply the mathematics to other situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an interview, or deviate from a known procedure to find a shortcut.”

STANDARDS FOR MATHEMATICAL PRACTICE (Continued)

Habits of Mind

MP1 **Make sense of problems and persevere in solving them**	**MP6** **Attend to precision**
○ Understand a problem and look for entry points ○ Consider given information, constraints, and relationships ○ Consider simpler or analogous problems ○ Make conjectures, monitor progress and alter their solution course as needed ○ Explain correspondences using multiple representations ○ Understand and analyze the approaches of others ○ Continually ask, "Does this make sense?"	○ Communicate mathematical ideas precisely ○ Use clear definitions ○ State meaning of symbols and use them properly ○ Attend to units of measures and labeling of axes ○ Calculate accurately and give solution with appropriate degree of accuracy

Reasoning and Explaining

MP2 **Reason abstractly and quantitatively**	**MP3** **Construct viable arguments and critique the reasoning of others**
○ Attend to the meaning of quantities ○ Decontextualize a problem using symbols, and manipulate them as if they have a life of their own ○ Contextualize manipulations to create a coherent representation of a problem	○ Use assumptions, definitions, and established results to create arguments (deductive reasoning) ○ Make and test conjectures based on evidence (inductive reasoning) ○ Analyze situations by breaking them into cases ○ Use counterexamples to disprove a statement ○ Identify flaws in an argument ○ Listen to or read to arguments and ask useful questions to clarify reasoning

STANDARDS FOR MATHEMATICAL PRACTICE (Continued)

<table>
<tr><th colspan="2"><u>Modeling and Using Tools</u></th></tr>
<tr><th>MP4
Model with Mathematics</th><th>MP5
Use appropriate tools strategically</th></tr>
<tr><td>o Apply mathematics to solve everyday problems
o Make reasonable assumptions and approximations to simplify a situation
o Identify important quantities in a situation
o Use multiple representations to analyze relationships and draw conclusions
o Interpret results in the context of the situation
o Improve the mathematical approach (model) if it has not served its purpose</td><td>o Select useful tools such as paper and pencil, graph paper ruler, calculator, concrete model, spreadsheet, or statistical software to solve problems
o Use concrete models and technology tools to explore concepts
o Recognize limitations of tools
o Identify and use relevant external resources, such as the internet</td></tr>
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<table>
<tr><th colspan="2"><u>Structure and Generalizing</u></th></tr>
<tr><th>MP7
Look for and make use of structure</th><th>MP8
Look for and make use of repeated reasoning</th></tr>
<tr><td>o Identify patterns and apply them to solve problems
o Recognize the structure of a symbolic representation and generalize it
o See complicated objects as composed of chunks of simpler objects</td><td>o Identify repeated calculations and patterns
o Generalize procedures based on repeated patterns or calculations
o Find shortcuts based on repeated patterns or calculations</td></tr>
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