HANDS-ON TRANSFORMATIONS: RIGID MOTIONS AND CONGRUENCE

(Poll Code 39934)

Presented by
Shelley Kriegler

President, Center for Mathematics and Teaching
shelley@mathandteaching.org

Fall 2014

STANDARDS FOR MATHEMATICAL CONTENT	
8.F.1	Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
8.G.1a	Verify experimentally the properties of rotations, reflections, and translations: Lines are taken to lines, and line segments to line segments of the same length.
8.G.1b	Verify experimentally the properties of rotations, reflections, and translations: Angles are taken to angles of the same measure.
8.G.1c	Verify experimentally the properties of rotations, reflections, and translations: Parallel lines are taken to parallel lines.
8.G.3	Describe the effect of ~~dilations~~, translations, rotations, and reflections on two-dimensional figures using coordinates.
F-IF-1	Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
G-CO-2	Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

STANDARDS FOR MATHEMATICAL PRACTICE	
MP2	Reason abstractly and quantitatively.
MP3	Construct viable arguments and critique the reasoning of others.
MP5	Use appropriate tools strategically.
MP7	Look for and make use of structure.

RIGID MOTIONS
TWO-SIDED TRANSFER TECHNIQUE

You will need a sheet of patty paper, a straightedge, and a pencil for this activity.

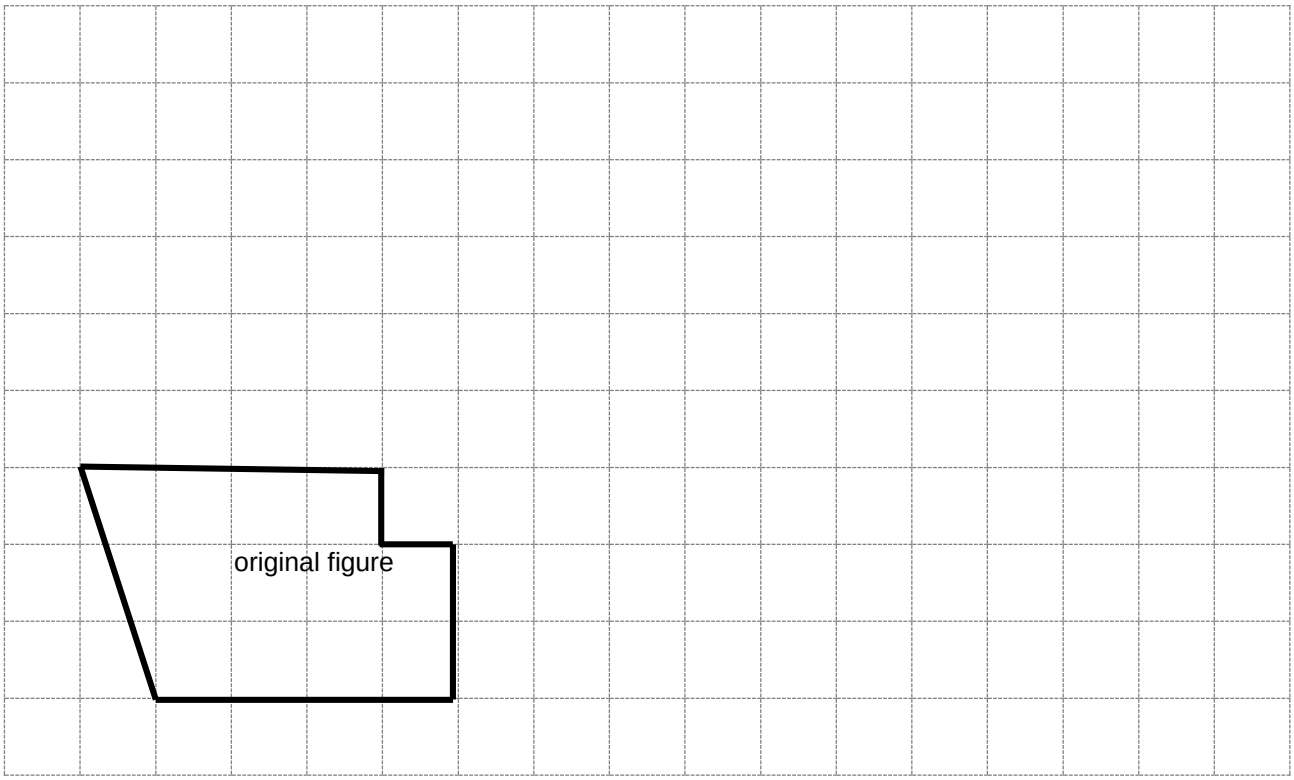
1. Trace the figure below on your patty paper. Make sure the edges of your patty paper line up with the grid lines below.

2. Turn the patty paper over to the backside. Use your pencil to trace over the figure from the front side of the patty paper.

3. Turn the patty paper over to the front side and put it on top of the original figure so that your marks coincide with the original figure. Shift the patty paper to the right, keeping your paper aligned with the horizontal gridlines. Then slide up, keeping your paper aligned with the vertical grid lines.

4. Trace over the patty paper figure. The pencil markings from the backside should transfer to this page. Follow the light pencil markings to draw the resulting figure.

5. Label the new figure as the "<u>image figure</u>."



We will call this drawing technique the "two-sided transfer technique."

TRANSFORMATIONS

On the previous page, you were given a figure. Using patty paper, you performed a "slide" to locate a new figure (called the image figure). The patty paper represents a plane, and the action represents a transformation.

1. Shade the original figure on the previous page and label the figure *ABCDEF*.

2. Do not shade the image figure. Label the figure $A'B'C'D'E'F'$. Be sure to mark corresponding vertices with the same letter.

> A transformation of the plane is a function that maps the plane to the plane.
>
> In other words, a transformation of the plane is a function that takes all points in the plane to points in the plane.

3. Refer to the transformation on the previous page. Complete each sentence. Notice how some of the language of transformations is used here.

   Point *A* is taken to Point _____. $\overline{DE}$ is taken to _____.

   Point *B* maps to Point _____. $\angle BCD$ maps to _____.

   Point C' is the image of Point _____. Figure $A'B'C'D'E'F'$ is the image of _____.

4. Your teacher will give you directions for locating a point P and its image P' on your diagram. How are they related to the original figure and its image?

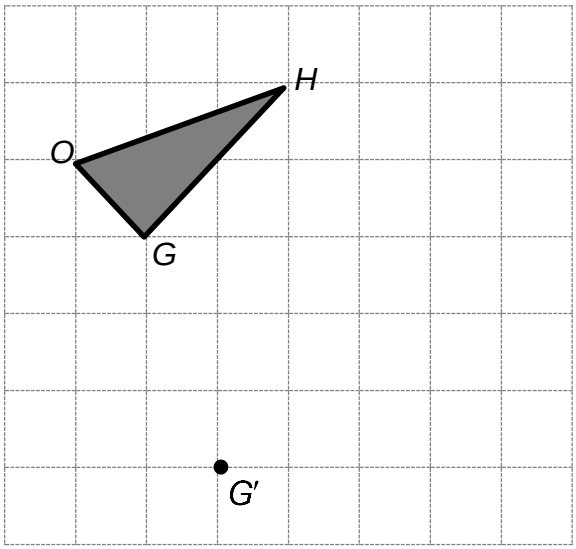
5. Sometimes a transformation results in an image figure that exactly covers the original figure. Do you think the transformation on the previous page resulted in an image figure that exactly covers the original figure? _____ Why?

TRANSLATIONS

A translation (or "slide") of the plane shifts all points the same distance and in the same direction.

- Draw the shape with pencil markings on both sides of the patty paper.
- Perform the translation (slide) using patty paper, and record using the two-sided transfer technique. Do not shade the image.

1. Translate $HOG \rightarrow H'O'G'$



2. Draw "arrows" to show the translation for each vertex of figure *HOG*. These arrows are called translation vectors. Label the translation vectors for $HOG \rightarrow H'O'G'$ as $\vec{v}$. What do you notice about the translation vectors?

We can create a function to show a translation. For example, the function $(x, y) \rightarrow (x + 2, y + 3)$ tells us that the image for every point (x, y) in the plane is obtained by adding 2 units to the x-coordinate and 3 units to the y-coordinate.

4. Write a function to describe the translation $HOG \rightarrow H'O'G'$ above.

5. Which of these properties are illustrated in the translation above?

   Segments taken to segments of the same length._____

   Angles are taken to angles of the same measure. _____

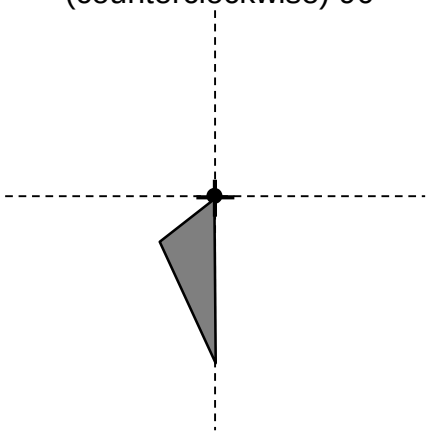
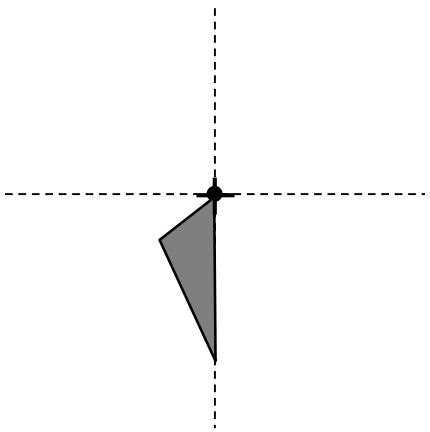
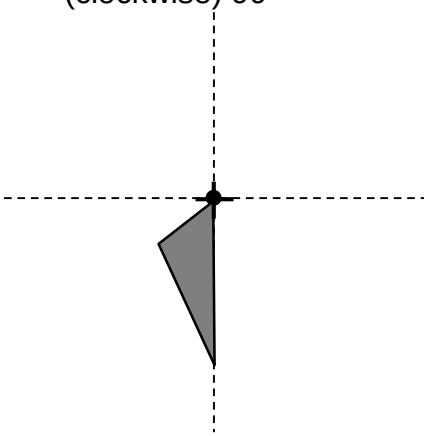
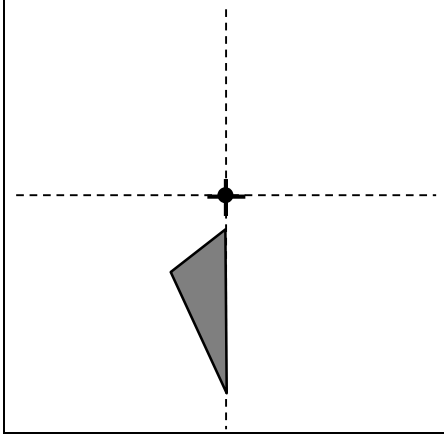
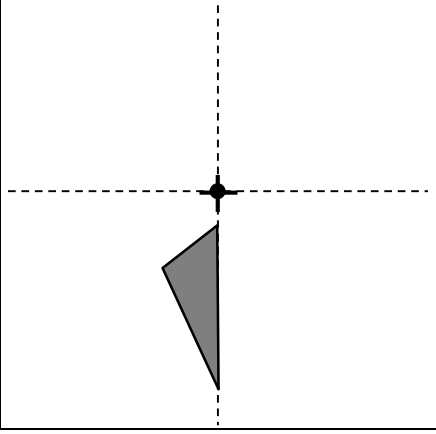
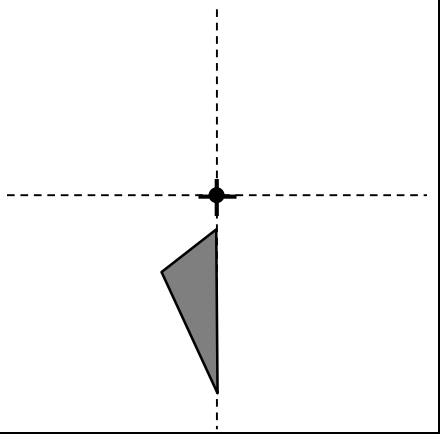
   Parallel lines are taken to parallel lines. _____

ROTATIONS

A rotation (or turn) is a transformation where a figure is turned through a given angle about a given point, called the center point of rotation. The point of rotation can be on the figure, inside the figure, or outside the figure.

For each row:

- Draw the shape on patty paper with pencil markings on both sides of the paper.
- Make a small ⊥ (with arrow pointing up) to indicate location of axes.
- Perform the rotation about the origin using patty paper and record.
- Do not shade the image.

1a. Rotate left (counterclockwise) 90° 	1b. Rotate left 180° 	1c. Rotate right (clockwise) 90° 
2a. Rotate right 90° 	2b. Rotate right 180° 	2c. Rotate left 90° 

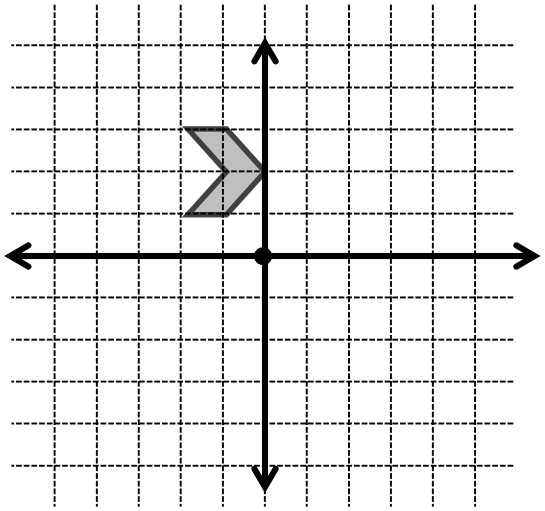
3. What happens to the center under a rotation?

4. What happens to every other point in the plane under a rotation?

ROTATION EXPLORATION

For each figure, rotate the plane 90º counterclockwise around the origin (0, 0).

1.



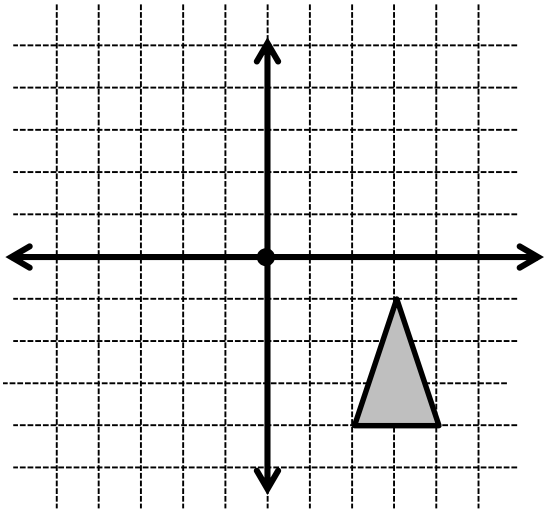
Some coordinates in the original figure

(____, ____) (____, ____) (____, ____)

Corresponding coordinates in its image

(____, ____) (____, ____) (____, ____)

2.



Some coordinates in the original figure

(____, ____) (____, ____) (____, ____)

Corresponding coordinates in its image

(____, ____) (____, ____) (____, ____)

3. Describe the relationship between coordinates in the original figure and its image using words.

4. Describe the relationship between coordinates in the original figure and its image using symbols.

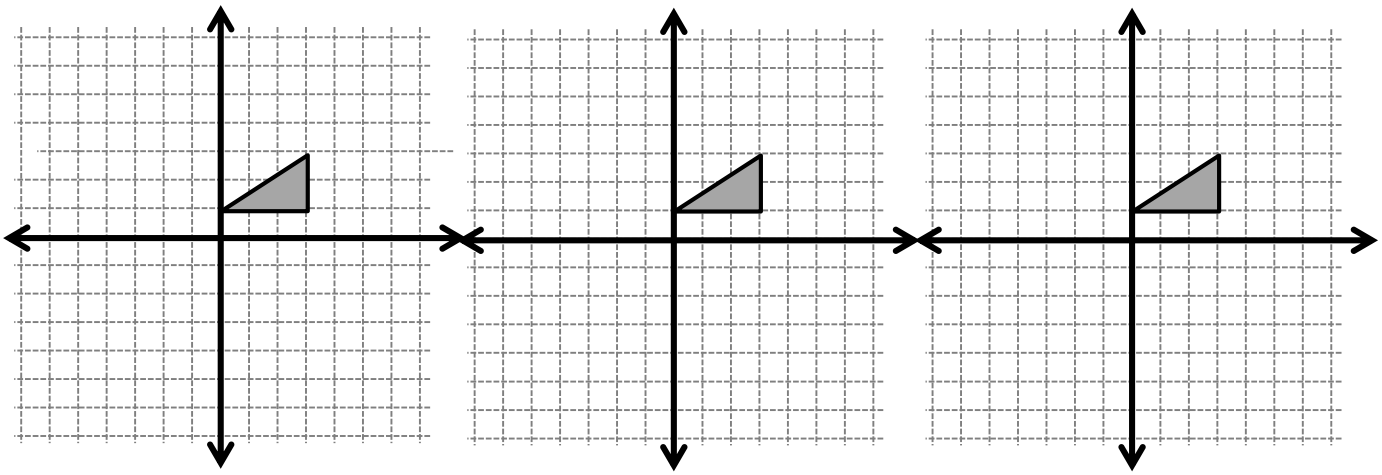
$(x, y) \rightarrow$ (_____, _____)

5. Use your work above to give an example showing that parallel lines are mapped to parallel lines under a rotation.

WHAT'S HAPPENING HERE?

- Triangle *ABC* has vertices at *A* (3, 1), *B* (3, 3), *C* (0, 1).
- Transfer these coordinates to the table at right prior to beginning the problems.
- Explore the effects of transformations (which are not necessarily translations, rotations, and reflections) by observing the effect on the image of the triangle.

Original Figure		
	x	*y*
A		
B		
C		



1. $(x, y) \rightarrow (-x, y - 2)$		
	-x	*y – 2*
A′		
B′		
C′		

2. $(x, y) \rightarrow (2x, 2y)$		
A′		
B′		
C′		

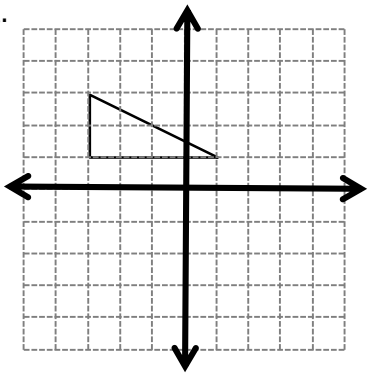
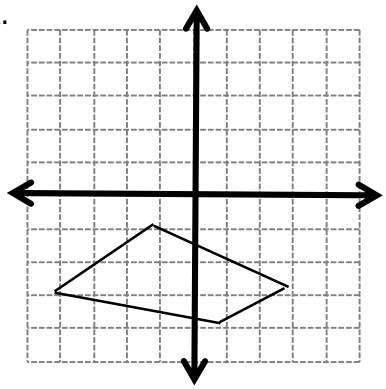
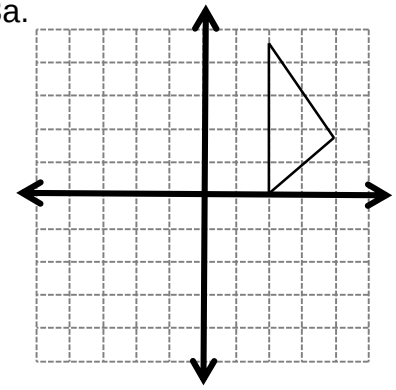
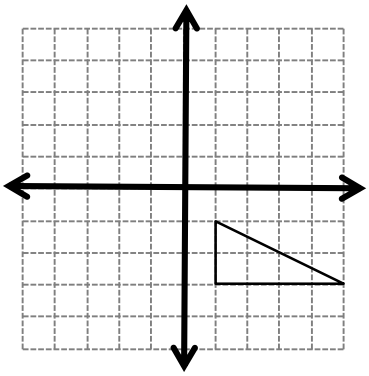
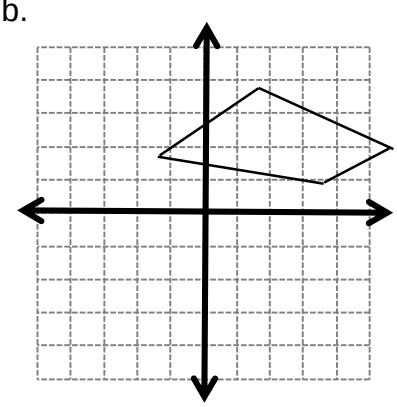
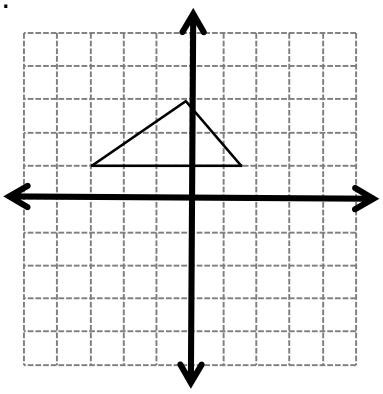
3. $(x, y) \rightarrow (x, x - y)$		
A′		
B′		
C′		

Refer to each of the three problems above (tables and diagrams).

	Is the transformation a translation, rotation, or reflection?	Are lines taken to lines?	Are segments taken to segments of the same length?	Are angles taken to angles of the same measure?
1.				
2.				
3.				

TALKING TRANSFORMATIONS

Complete this activity with a partner following the instructions of your teacher.

1a.  Steps:	2a.  Steps:	3a.  Steps:
1b.  Steps:	2b.  Steps:	3b.  Steps:

Two figures in the plane are <u>congruent</u> if the second can be obtained from the first by a sequence of translations, rotations, and reflections.

4. How do you know that the figures and images in the "Talking Transformations" activity are congruent?

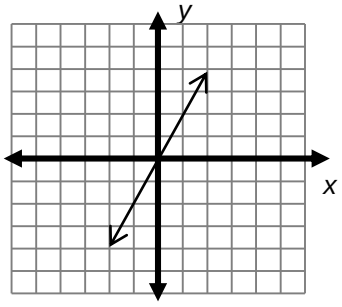
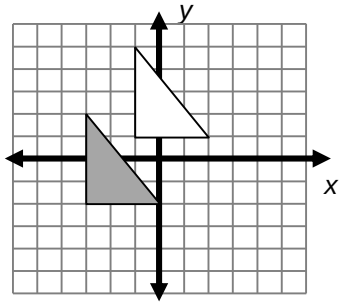
MATH NOTE

The Vocabulary of Geometry

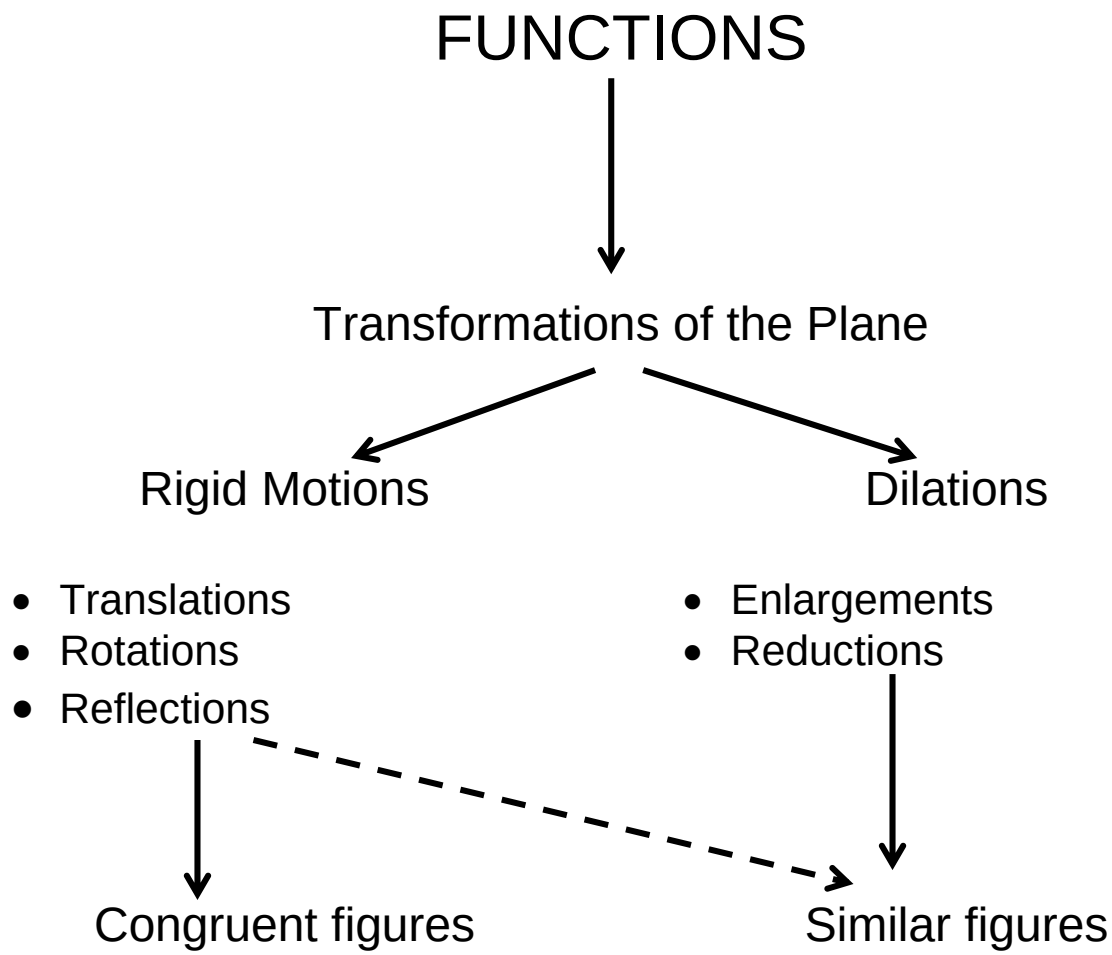
The function concept is one of the most important and widespread concepts in mathematics, When the idea comes up in a specific field, the vocabulary is often adapted to give the flavor of the field. In probability and statistics, the functions of importance are defined on a sample space, and they are referred to as "random variables." In functional analysis, the functions from one vector space to another are referred to as "operators" or "transformations." In geometry, a function from one space to another is often referred to as a "transformation" or as a "map," and we talk about "mapping" one space to another.

How we visualize functions graphically also varies from field to field. In probability and statistics, what is important is not the specific random variable but rather its "distribution" function, which describes how the values of the random variable are distributed probabilistically. Everyone is familiar with the bell-shaped curve of the normal distribution function, which represents the distribution of values of a normal random variable. In school algebra we may visualize a function by representing its graph in a coordinate plane, graphing the output value (y-value) against the input value (x-value). In geometry we may visualize functions by drawing arrows, pointing from points of the domain space to their image points in the range space. We may also visualize the function by sketching how the function transforms certain figures. This helps us to understand whether the function is a translation, rotation, or reflection, or whether the function is some sort of stretching or other operation.

The differences in describing and visualizing the functions of school algebra and geometry are portrayed in the following table, contrasting a typical algebra function on the line and a typical transformation of the plane.

verbal description	a rule that assigns each real number to twice its value	a rule that shifts all points in the plane 2 units to the right and 3 units up
symbolic notation	x maps to $2x$ $f(x) = 2x$ $x \rightarrow 2x$	(x, y) maps to $(x + 2, y + 3)$ $T(x, y) = (x + 2, y + 3)$. $(x, y) \rightarrow (x + 2, y + 3)$
graph		
graph interpretation	The x-coordinates represent the inputs and the y-coordinates represent the outputs. The set of input-output pairs is represented by the line.	A figure (shaded triangle) and its image (unshaded triangle) illustrate what is happening to points in the plane.

The Common Core State Standards postpone the introduction of function notations such as $f(x)$ and $T(x,y)$ to the high school curriculum. Consequently we make heavy use of the arrow notation to describe transformations in 8th grade.


FUNCTIONS
Transformations of the Plane
Rigid Motions
Dilations
• Translations
• Rotations
• Reflections
• Enlargements
• Reductions
Congruent figures
Similar figures

Evaluate this session online. Follow these directions.