HANDS-ON TRANSFORMATIONS: DILATIONS AND SIMILARITY

(Poll Code 44273)

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STANDARDS FOR MATHEMATICAL CONTENT	
8.F.1	Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
8.G.3	Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
8.G.4	Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two dimensional figures, describe a sequence that exhibits the similarity between them.
8.G.6	Explain a proof of the Pythagorean Theorem and its converse.
F-IF-1	Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then *f*(*x*) denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation *y* = *f*(*x*).
G-CO-2	Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
G-SRT1a	Verify experimentally the properties of dilations given a center and scale factor: A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves the line passing through the center unchanged.

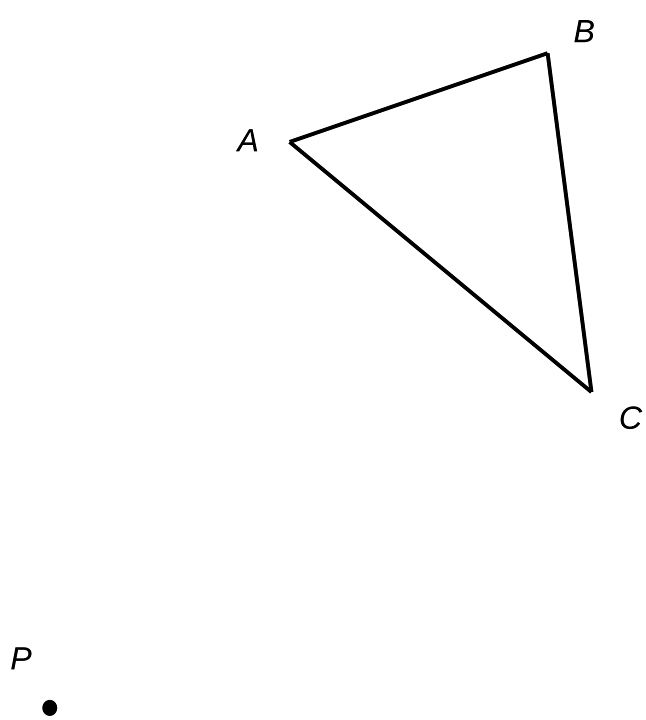
DILATIONS

RUBBER BAND EXPERIMENT

Use SMALL rubber bands. Link two of them together. Anchor one end of the band at point *P* with a pencil. Put another pencil into the other end of the band. Move this pencil so that the knot in the middle of the band traces over the triangle. You will create a new figure that is the image of the original under a transformation.

Shade the original triangle. Do not shade the image. "Clean up" your image by drawing segments formed by the rubber band transformation with a ruler. Label the corresponding vertices *A*′, *B*′, *C*′.

What is the shape of the image?



RUBBER BAND EXPERIMENT (Continued)

Refer to the diagram you created on the previous page. Make a copy of ΔABC on patty paper or cut out ΔABC at the bottom of this page to help you answer these questions.

1. Is $\Delta ABC \cong \Delta A'B'C'$? Explain.

2. How are the corresponding angles of ΔABC related to the corresponding angles in $\Delta A'B'C'$? How do you know?

3. The area of $\Delta A'B'C'$ is about how many times as large as the area of ΔABC?

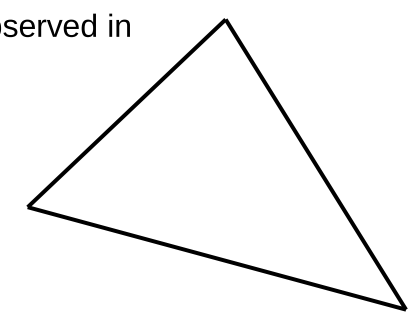
4. How are corresponding sides of ΔABC related to the corresponding sides in $\Delta A'B'C'$? How do you know?

5. Find these ratios: $\frac{|A'B'|}{|AB|} =$ ______ $\frac{|B'C'|}{|BC|} =$ ______ $\frac{|A'C'|}{|AC|} =$ ______

6. How are these ratios related to the rubber band experiment?

7. Draw $\overrightarrow{PA'}$. Where is point A in relation to $\overrightarrow{PA'}$? Explain why this happened.

8. Draw $\overrightarrow{PB'}$ and $\overrightarrow{PC'}$. Did the relationships you observed in problem 7 above occur again?

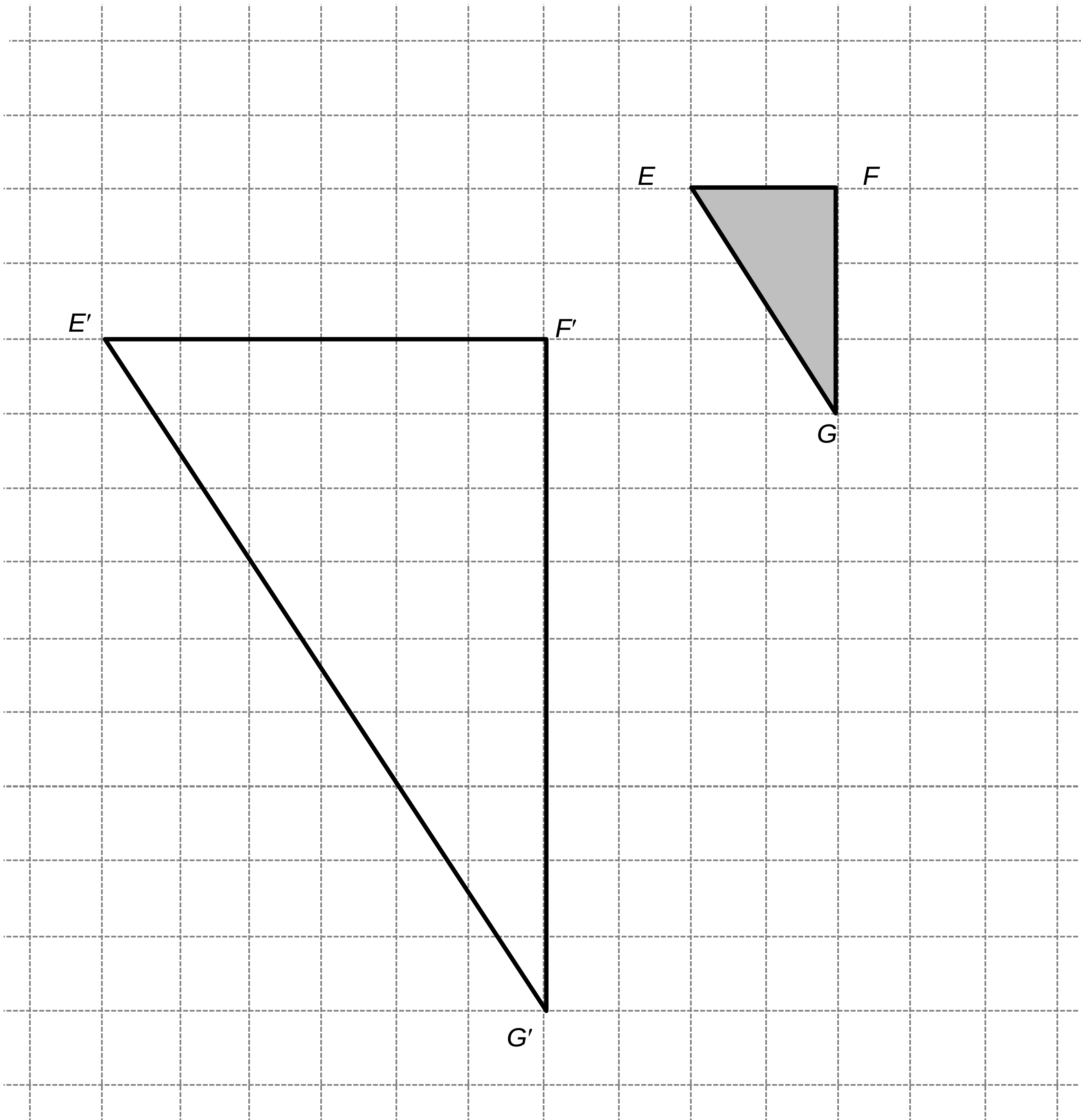


GRAPH PAPER EXPERIMENT 1

ΔEFG maps to $\Delta E'F'G'$.

1. Find the lengths of the sides of each triangle.

2. Explain why the triangles are not congruent.



GRAPH PAPER EXPERIMENT 1 (Continued)

3. Are corresponding angles congruent? Explain.

4. What is the relationship between corresponding sides? Be sure to check all sides and then explain.

5. Draw lines between corresponding vertices in the triangles. Extend the lines so they intersect each other. What do you notice about the point(s) of intersection?

6. Label Q as the point of intersection. Find the following ratios.

$$\frac{|E'Q|}{|EQ|} = \_\_\_\_\_ \qquad \frac{|F'Q|}{|FQ|} = \_\_\_\_\_ \qquad \frac{|G'Q|}{|GQ|} = \_\_\_\_\_$$

We call this ratio the <u>scale factor</u>.

7. How are the ratios formed from these segments related to the relationship you observed between corresponding sides?

8. Are corresponding sides in the figure and its image parallel? Explain.

DILATIONS

The previous experiments illustrate a transformation called a dilation.

A dilation is a transformation that moves each point along the ray through the point emanating from a fixed center, multiplying distances from the center by a common scale factor.

1. Name some features of dilations on the previous pages.

	$\Delta ABC \rightarrow \Delta A'B'C'$	$\Delta EFG \rightarrow \Delta E'F'G'$
fixed center		
rays through the center		
scale factor		

2. Under the dilations you have observed, are line segments taken to line segments of the same length?

3. Under the dilations you have observed, are angles taken to angles of the same measure?

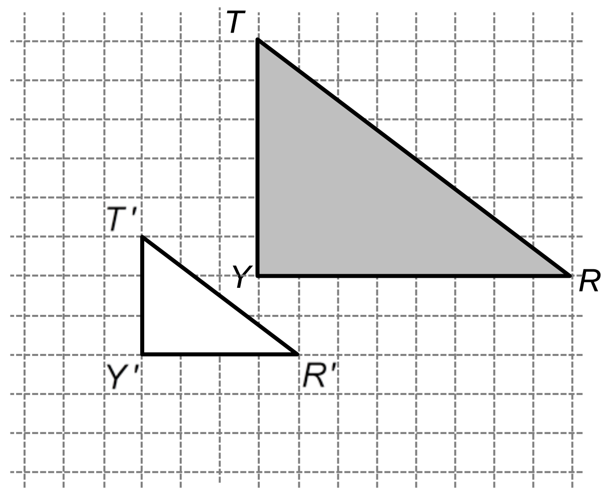
4. How are dilations different from translations, rotations, and reflections?

Inductive reasoning is a form of reasoning in which the conclusion is supported by the evidence but is not proved.

5. What are some properties you have observed about dilations, based on the evidence of the experiments?

DILATIONS IN THE COORDINATE PLANE

$\Delta T'R'Y'$ is the image of ΔTRY under a dilation.



1. Find the lengths of the sides of the triangles. Label them on the figures.

2. Use a ruler to draw lines TT', RR' and YY'. Label their intersection as point *C.*

   Point *C* is the ________ __________ of the dilation.

3. Draw an *x*-axis and *y*-axis through the center point. What are the coordinates of *C*? (____, ____)

4. What are the coordinates of the vertices in the original figure (shaded) and the image (unshaded)?

   T (_____, _____) *R* (_____, _____) *Y* (_____, _____)

   T'(_____, _____) ***R'***(_____, _____) ***Y'***(_____, _____)

5. Describe a relationship between the coordinates in the original figure and coordinates in the image.

6. Use symbols to describe the how the original figure maps to its image.

   (*x* , *y*) → (______, ______)

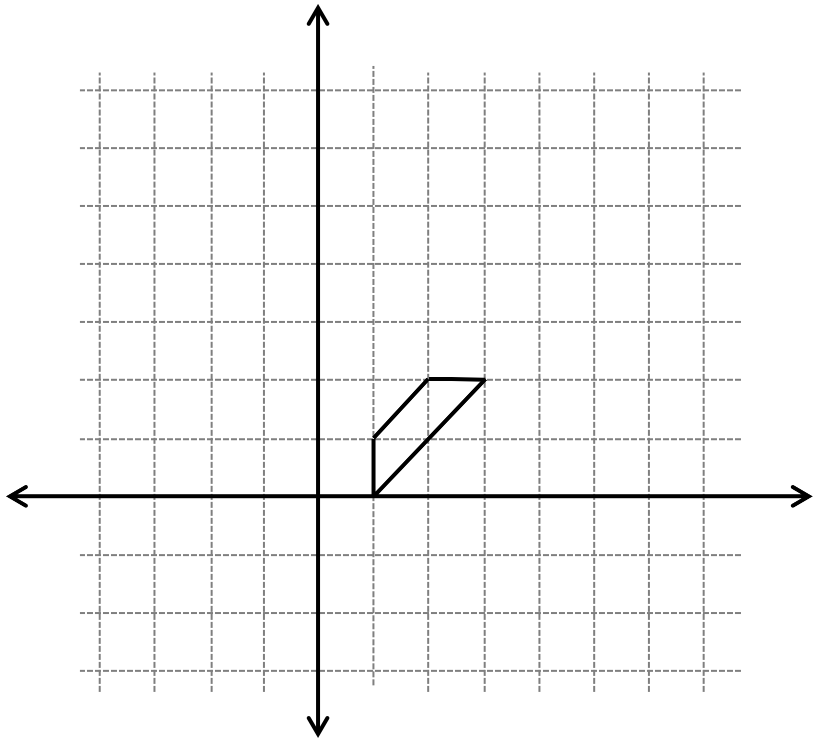
7. What is the scale factor of the figure. $\frac{|CT'|}{|CT|} =$ ______ $\frac{|CR'|}{|CR|} =$ ______ $\frac{|CY'|}{|CY|} =$ ______

8. How is the scale factor different in this example from those in previous examples?

9. What does that mean in terms of the figure and its image?

To dilate a figure in the coordinate plane with the center of dilation at the origin, multiply the coordinates of its points by the scale factor.

PROPERTIES OF TRANSFORMATIONS



Use coordinates and diagrams to show the image of each transformation.

T (x, y)	*T* → *f*(*T*) (*x*, *y*) → (-*x*, *y*)	*T* → *g*(*T*) (*x*, *y*) → (2*x*, 2*y*)	*T* → *h*(*T*) (*x*, *y*) → (x + y, -y)
A(1, 0)			
B(1, 1)			
C(2, 2)			
D (3, 2)			
Name of transformation			

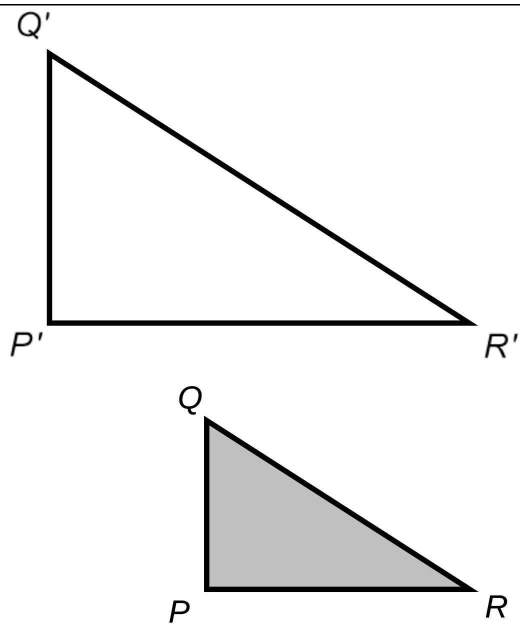
Answer “yes” or “no” for each transformation above.

		f(*T*)	*g*(*T*)	*h*(*T*)
3.	Is the image of the figure congruent to the original figure?			
4.	Is the transformation a dilation?			
5.	Are segments taken to segments of equal length?			
6.	Are angles taken to angles of equal measure?			
7.	Are parallel lines taken to parallel lines?			

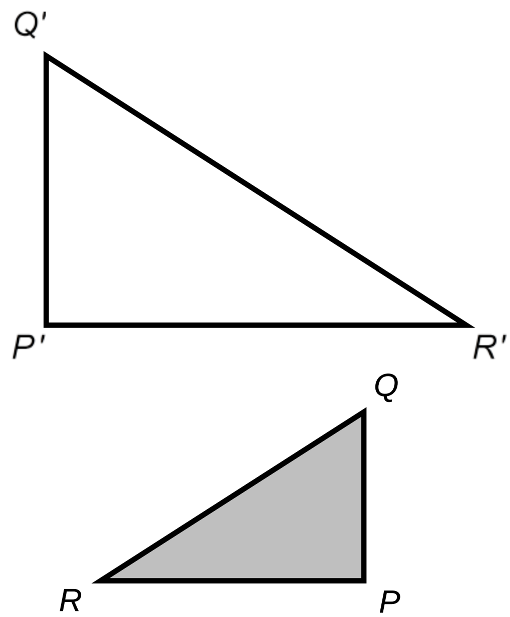
SIMILARITY

Two figures are similar if one can be obtained from the other by a sequence of translations, rotations, reflections, and dilations.

1.



2.



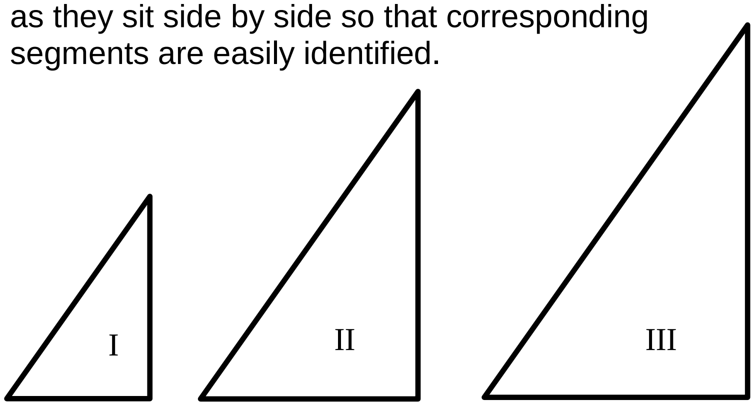
Deductive reasoning is a form of reasoning in which the conclusion is justified by an argument based on definitions, known facts, and accepted rules of logic.

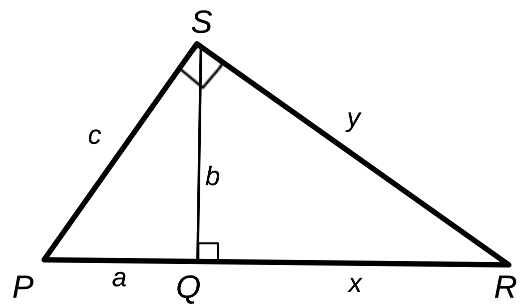
Decide if each statement is true or false. If it is true, justify your answer with deductive reasoning. If it is false, explain and give a counterexample.

3. Whenever two figures are congruent, they are similar.	4. Whenever two figures are similar, they are congruent.

SIMILARITY SURPRISE

1. Within the diagram at the right are three right triangles. Label angles and lengths of the triangles as they sit side by side so that corresponding segments are easily identified.





2. Establish that the triangles are similar using the AA Similarity Criterion.

Δ I ~ Δ III because ∠______ ≅ ∠______ and ∠______ ≅ ∠______ .

Δ II ~ Δ III because ∠______ ≅ ∠______ and ∠______ ≅ ∠______ .

Δ I ~ Δ II because ____ .

When triangles are similar, their sides are proportional.

3. Write an equation that states that $\dfrac{\text{length of shorter leg}}{\text{length of longer leg}}$ in triangles I and II are proportional. This proportion tells us that ax = _____	4. Write an equation that states that $\dfrac{\text{length of hypotenuse}}{\text{length of shorter leg}}$ in triangles I and III are proportional. This proportion tells us that ax = ________

5. Since two expressions are equal to *ax*, they are equal to each other. Write the equality. Then rewrite it so there are no negative coefficients.

What did you just prove?

MATH NOTES

Approaching Congruence and Similarity through Transformational Geometry

One of the most transparent content changes in the 2010 Common Core State Standards in Mathematics is the introduction of transformational geometry in 8^{th} grade, leading towards a transformational approach to geometry in high school. Whereas classical synthetic geometry focuses on geometric constructions as in Euclid's *Elements*, transformational geometry focuses on certain classes of geometric transformations and the properties of figures that remain invariant under them.

The 8th grade standards focus on two classes of transformations of the plane (functions from the plane to the plane).

The first important class of transformations of the plane is the <u>rigid motions</u>, or <u>isometries</u>, which are the transformations that preserve distance. Any rigid motion can be represented as a sequence of translations, rotations, and reflections. Rigid motions map lines to lines, and they preserve lengths, areas, and measures of angles. In transformational geometry, two figures are defined to be <u>congruent</u> if one can be mapped onto the other by a rigid motion.

The second important class of transformations of the plane is the <u>similarity transformations</u>, which are compositions of rigid motions and dilations. Similarity transformations map lines to lines, and they preserve angles. However, in general they do not preserve lengths and areas. In transformational geometry, two figures are defined to be <u>similar</u> if one can be mapped onto the other by a similarity transformation.

The sequence of concepts studied that relate to geometry of the plane in 8^{th} grade is anchored in the core idea of function and leads to the properties of congruence and similarity. It can be described as follows:

functions → transformations of the plane ↗ rigid motions (translations, rotations, reflections) → congruent figures

functions → transformations of the plane ↘ rigid motions and dilations (enlargements, reductions) → similar figures

The use of transformations as the foundation of geometry stems from the "Erlanger Programm" of Felix Klein. He laid the program out in an address he gave in 1872 upon becoming a professor of mathematics at the University of Erlangen in Germany. At the time he was 23 years old. He eventually moved to Göttingen and helped make it one of the most important centers for mathematics of his day. Around 1900 Klein became involved with math education, among other things pushing for introduction of the function concept and calculus ideas earlier in the German secondary math curriculum. Klein had an important effect on mathematics and math education in the United States, through the Americans who went to Germany to study at Göttingen, and through his students who took positions in the United States.

MATH NOTES (Continued)

Properties of Dilations

Dilations share many, but not all, of the properties of rigid motions. Some of the most important properties of dilations are:

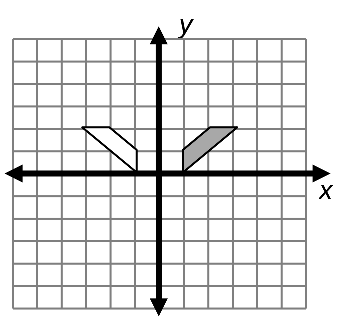
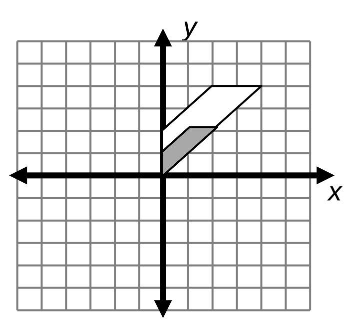
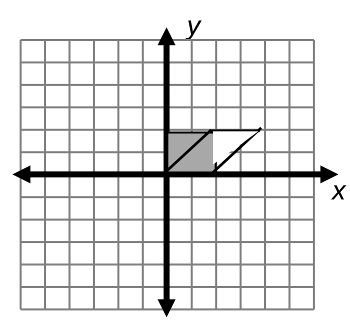
1. Dilations map lines to lines. The way to see this is to introduce a coordinate system with origin at the center of the dilation. With respect to coordinates, the dilation then takes the form $(x, y) \rightarrow (sx, sy)$, where $s > 0$ is the scale factor of the dilation. We calculate that the image of the line with the equation $y = mx + b$ is the line with the equation $y = mx + sb$. Thus the image of a line under a dilation is a line with the same slope.

2. Dilations preserve parallelism. Indeed, if two lines do not meet, then their images under a dilation do not meet.

3. Dilations preserve angle measure. Consider an angle at a point P, determined by two lines intersecting at P. The dilation maps each of these lines to a parallel line through the image Q of P. Since the lines through Q are parallel to the lines through P, the angles they determine at Q have the same measure as the corresponding angles of the lines through P, and by checking which angle corresponds with which, one concludes that the angles at P have the same measures as their image angles at Q.

4. In general, dilations do not preserve distances. The only dilation (with scale factor $s > 0$) that preserves distances is the dilation with scale factor $s = 1$, that is, the identity transformation $(x, y) \rightarrow (x, y)$.

Examples and non-examples

$(x, y) \rightarrow (-x, y)$	$(x, y) \rightarrow (2x, 2y)$	$(x, y) \rightarrow (x + y, y)$
		
Rigid motion (reflection), preserves lines, angle measures, and distances, and also lengths and areas.	Dilation, preserves lines and angle measures, but not distances.	Shear, preserves lines, but neither angle measures nor distances.